THE SECOND HANKEL DETERMINANT FOR A CLASS OF λ-q-SPIRALLIKE FUNCTIONS

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Abstract. The object of the present paper is to obtain an upper bounded to the second Hankel determinant $|a_2a_4 - a_2^3|$ for λ-q- spirallike function of f .

Keywords: Analytic function, λ-q-spirallike functions, Upper bound, Second Hankel determinant.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n, \tag{1}$$

defined on the unit disk $E = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$ normalized by $f(0) = 0$, $f'(0) = 1$. Let S denote the subclass of function in $\mathcal{A}$ which are univalent in E. The Hankel determinants of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [22], as

$$\mathbf{H_q(n)} := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}$$

where $(n = 1, 2, ... \; and \; q = 1, 2, ...,)$. This determinant has been considered by several authors in the literature.

For example, Noonan and Thomas[34] studied about the second Hankel determinant of a really mean p-valent functions. Noor [21], determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. Ehrenborg [13], studied the Hankel determinant of exponential polynomials.

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Definition 1.1. [11] *The q-analogue of f is given by*

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad , \quad where (0 < q < 1) \tag{2}$$

Equivalently (2), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{q} \neq 1 \\ n, & \text{q} = 1 \end{cases}$$

Note that as $q \to 1$, $[n]_q \to n$.

Definition 1.2. *A function $f \in A$ is said to be λ-q-spiral starlike ($|\lambda| \leq \frac{\pi}{2}$), if and only if*

$$\Re \left\{ e^{i\lambda} \frac{z \partial_q f(z)}{f(z)} \right\} \geq 0, z \in E. \tag{3}$$

The class of λ-spiral starlike functions defined and studied by Spacek [] is denoted by $SPST(\lambda)$. In this paper we study the class of λ-q-spiral starlike functions and denoted by $SPST(\lambda, q)$. It is observed when $\lambda = 0, SPST(0, q) = ST_q$.

Definition 1.3. *A function $f \in A$ is said to be convex λ-q-spiral, where$\left(\frac{-\pi}{2} \leq \lambda \leq \frac{\pi}{2}\right)$, if it satisfies the condition*

$$\Re \left[e^{i\lambda} \left\{ \frac{z \partial_q^2 f(z)}{\partial_q f(z)} \right\} \right] \geq 0, z \in E. \tag{4}$$

The class of convex λ-spiral functions defined by Robertson (according to Goodman[]) is denoted by $CVSP(\lambda)$. In this paper we study the class of convex λ-q-spiral functions and denoted by $CVSP(\lambda, q)$. It is observed when $\lambda = 0, CVSP(0, q) = CV_q$.

Let $\mathcal{P}$ denote the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = \left\{ 1 + \sum_{n=1}^{\infty} c_n z^n \right\}, \forall z \in E. \tag{5}$$

Lemma 1.1. [4] *If the function $p \in \mathcal{P}$ is given by the series* (5) *then the following sharp estimate holds:*

$$|c_n| \leq 2 \quad (n = 1, 2, ...).$$

Lemma 1.2. [8] *If the function $p \in \mathcal{P}$ is given by the series* (5), *then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{6}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{7}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Theorem 1.1. *If* $f(z) = z + \sum_{n=2}^{\infty} \in SPST(\lambda, q), (|\lambda| < \frac{\pi}{3})$ *then*

$$|a_2a_4 - a_3^2| \;\le\; \frac{4\cos^2\lambda}{([3]_q - 1)^2}.$$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} \in SPST(\lambda, q)$, from (1), there exists an analytic function $p \in \mathcal{P}$ in the unit disc E with $p(0) = 1$ and $\Re\ \{p(z)\} > 0$ such that

$$e^{i\lambda}\left\{\frac{z\partial_q f(z)}{f(z)}\right\} = p(z) \Rightarrow \left\{e^{i\lambda}z\partial_q f(z) - i\sin\lambda f(z)\right\} \tag{8}$$

$$= \cos\lambda\left\{f(z) \times p(z)\right\}.$$

Replacing $f(z), \partial_q f(z)$ and $p(z)$ with their equivalent series expressions in (8), we have

$$\left[e^{i\lambda}\left\{1 + \sum_{n=2}^{\infty}[n]_q\ a_n\ z^{n-1}\right\} - i\sin\lambda\left\{z + \sum_{n=2}^{\infty}\ a_n\ z^n\right\}\right]$$

$$= \cos\lambda\left[\left\{z + \sum_{n=2}^{\infty}\ a_n\ z^n\right\} \times \left\{1 + \sum_{n=2}^{\infty}\ c_n\ z^n\right\}\right].$$

Upon simplification, we obtain

$$e^{i\lambda}\left[([2]_q - 1)a_2z + ([3]_q - 1)a_3z^2 + ([4]_q - 1)a_4z^3 + ...\right] \tag{9}$$

$$= \cos\lambda\left[c_1z + (c_2 + c_1a_2)z^2 + (c_3 + c_2a_2 + c_1a_3)z^3 + ...\right].$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively in (9), we have
$\big[([2]_q - 1)a_2e^{i\lambda} = c_1\cos\lambda, \qquad ([3]_q - 1)a_3e^{i\lambda} = (c_2 + c_1a_2)\cos\lambda,$
$([4]_q - 1)a_4e^{i\lambda} = (c_3 + c_2a_2 + c_1a_3)\cos\lambda\big]$.
After simplifying, we get

$$\left[a_2 = \frac{e^{-i\lambda}c_1\cos\lambda}{[2]_q - 1}, \qquad a_3 = \frac{1}{([3]_q - 1)([2]_q - 1)}\left\{([2]_q - 1)c_2 + c_1^2e^{-i\lambda}\cos\lambda\right\}\cos\lambda,\right.$$

$$\left. a_4 = \frac{e^{-i\lambda}}{([4]_q - 1)([3]_q - 1)([2]_q - 1)}\left\{([3]_q - 1)([2]_q - 1)c_3 + F\right\}\cos\lambda\right], \tag{10}$$

where $F = \left(([3]_q - 1) + ([2]_q - 1)\right)c_1c_2e^{-i\lambda}\cos\lambda + c_1^3e^{-2i\lambda}\cos^2\lambda$.
Substituting the values of a_2, a_3, and a_4 from(10) in the second Hankel functional $|a_2a_4 - a_3^2|$ for the function $f \in SPST(\lambda, q)$, we have

$$|a_2a_4 - a_3^2| = \left|\frac{e^{-i\lambda}c_1\cos\lambda}{([2]_q - 1)^2} \times \frac{e^{-i\lambda}}{([4]_q - 1)([3]_q - 1)}\left\{([3]_q - 1)([2]_q - 1)c_3 + F\right\}\cos\lambda\right.$$

$$\left. - \frac{e^{-2i\lambda}}{([3]_q - 1)^2([2]_q - 1)^2}\left\{([2]_q - 1)c_2 + c_1^2e^{-i\lambda}\cos\lambda\right\}^2\cos^2\lambda\right|,$$

where $F = \left(([3]_q - 1) + ([2]_q - 1)\right)c_1c_2e^{-i\lambda}\cos\lambda + c_1^3e^{-2i\lambda}\cos^2\lambda$.

Using the facts $|xa+yb| \leq |x||a|+|y||b|$, where x, y, a and b are real numbers and $|e^{-in\lambda}|=1$, where n is a real number, upon simplification, we obtain

$$|a_2a_4-a_3^2| \leq \frac{\cos^2\lambda}{([4]_q-1)([3]_q-1)^2([2]_q-1)^2}\left|([3]_q-1)^2([2]_q-1)c_1c_3-R\right|, \quad (11)$$

where $R=([4]_q-1)([2]_q-1)^2c_2^2-(([4]_q-1)-([3]_q-1))\,c_1^4\cos^2\lambda$.
Substituting the values of c_2 and c_3 from (6) and(7) respectively from Lemma 1.2 on the right-hand side of (11), we have

$$\begin{aligned}&\left|([3]_q-1)^2([2]_q-1)c_1c_3-([4]_q-1)([2]_q-1)^2c_2^2-(([4]_q-1)-([3]_q-1))\,c_1^4\cos^2\lambda\right|\\ &=\Bigg|([2]_q-1)([3]_q-1)^2c_1\times\frac{1}{4}\left\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\right\}\\ &-\frac{1}{4}\times([4]_q-1)([2]_q-1)^2\left\{c_1^2+x(4-c_1^2)\right\}^2-(([4]_q-1)-([3]_q-1))\,c_1^4\cos^2\lambda\Bigg|.\end{aligned}$$

Using the fact $|z|<1$, upon simplification, we obtain

$$\begin{aligned}&4\left|([3]_q-1)^2([2]_q-1)c_1c_3-([4]_q-1)([2]_q-1)^2c_2^2-(([4]_q-1)-([3]_q-1))\,c_1^4\cos^2\lambda\right|\\ &\leq\Big|\Big\{\big(([2]_q-1)([3]_q-1)^2-([2]_q-1)^2([4]_q-1)\big)-4\,(([4]_q-1)-([3]_q-1))\cos^2\lambda\Big\}\,c_1^4\\ &+2([2]_q-1)([3]_q-1)^2c_1(4-c_1^2)+2\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c_1^2(4-c_1^2)|x|\\ &-\left[\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c_1+2([2]_q-1)^2\right]H|x|^2\Big|,\end{aligned}$$

where $H=\left[\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c_1+2([4]_q-1)\right](4-c_1^2)$.
Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\geq(c_1-a)(c_1-b)$, where $a,b\geq 0$ on the right-hand side of the above inequality, we get

$$\begin{aligned}&4\left|([3]_q-1)^2([2]_q-1)c_1c_3-([4]_q-1)([2]_q-1)^2c_2^2-(([4]_q-1)-([3]_q-1))\,c_1^4\cos^2\lambda\right|\\ &\leq\Big|\Big\{\big(([2]_q-1)([3]_q-1)^2-([2]_q-1)^2([4]_q-1)\big)-4\,(([4]_q-1)-([3]_q-1))\cos^2\lambda\Big\}\,c_1^4\\ &+2([2]_q-1)([3]_q-1)^2c_1(4-c_1^2)+2\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c_1^2(4-c_1^2)|x|\\ &-\left[\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c_1-2([2]_q-1)^2\right]M|x|^2\Big|,\end{aligned} \quad (12)$$

where $M=\left[\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c_1-2([4]_q-1)\right](4-c_1^2)$.
Choosing $c_1=c\in[0,2]$, applying triangle inequality replacing $|x|$ by μ on the right-hand side of (12) we obtain

$$\begin{aligned}&4\left|([3]_q-1)^2([2]_q-1)c_1c_3-([4]_q-1)([2]_q-1)^2c_2^2-(([4]_q-1)-([3]_q-1))\,c_1^4\cos^2\lambda\right|\\ &\leq\Big|\Big\{4\,(([4]_q-1)-([3]_q-1))\cos^2\lambda-\big(([2]_q-1)([3]_q-1)^2-([2]_q-1)^2([4]_q-1)\big)\Big\}\,c^4\\ &+2([2]_q-1)([3]_q-1)^2c(4-c^2)+2\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c^2(4-c^2)\mu\\ &+\left[\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c-2([2]_q-1)^2\right]N\mu^2\Big|,\\ &=F(c,\mu), with\ 0\leq\mu=|x|\leq 1.\end{aligned} \quad (13)$$

where $N=\left[\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c-2([4]_q-1)\right](4-c^2)$.

$$\begin{aligned}F(c,\mu)&=\Big|\Big\{4\,(([4]_q-1)-([3]_q-1))\cos^2\lambda-\big(([2]_q-1)([3]_q-1)^2-([2]_q-1)^2([4]_q-1)\big)\Big\}\,c^4\\ &+2([2]_q-1)([3]_q-1)^2c(4-c^2)+2\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c^2(4-c^2)\mu\\ &+\left[\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c-2([2]_q-1)^2\right]N\mu^2\Big|,\end{aligned} \quad (14)$$

where $N = [(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2)\,c-2([4]_q-1)]\,(4-c^2)$.
Now the function $F(c,\mu)$ is maximized on the closed square $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (14) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \left[2\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c^2 \right.$$
$$\left. +2\mu\left\{\left(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2\right)c-2([2]_q-1)^2\right\}L\right]\times(4-c^2), \tag{15}$$

where $L = [(([2]_q-1)([3]_q-1)^2-([4]_q-1)([2]_q-1)^2)\,c-2([4]_q-1)]$.
For $0<\mu<1$, for fixed c,q with $0<c<2$ and $0<q<1$, from (15), we observe that $\frac{\partial F}{\partial \mu}>0$. Consequently, $F(c,\mu)$ is an increasing function of μ and hence cannot have maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c). \tag{16}$$

Upon simplifying the relations(14) and(16) we obtain

$$G(c) = \left[4\left\{(([4]_q-1)-([3]_q-1))\cos^2\lambda-\left\{\left(([2]_q-1)([3]_q-1)^2-([2]_q-1)^2([4]_q-1)\right)\right\}\right\}c^4 \right.$$
$$\left. +16\left(([2]_q-1)^2([4]_q-1)\right)\right], \tag{17}$$

$$G'(c) = \left[4[4]_q\left\{(([4]_q-1)-([3]_q-1))\cos^2\lambda-\left\{\left(([2]_q-1)([3]_q-1)^2-([2]_q-1)^2([4]_q-1)\right)\right\}\right\}c^3\right] \tag{18}$$

From the expression(18), we observe that $G'(c)\le 0$ for all values of c in the interval $0\le c\le 2$ and for a fixed value of λ with $\left(\frac{-\pi}{3}\le\lambda\le\frac{\pi}{3}\right)$. Therefore, $G(c)$ is a monotonically decreasing function of c in the interval $[0,2]$ so that its maximum value occurs at $c=0$. From(17), we get

$$\max_{0\le c\le2} G(0) = 16\left(([2]_q-1)^2([4]_q-1)\right) \tag{19}$$

After simplifying the expression (13) and (19), we obtain

$$\left|([3]_q-1)^2([2]_q-1)c_1c_3-([4]_q-1)([2]_q-1)^2c_2^2-(([4]_q-1)-([3]_q-1))\,c_1^4\cos^2\lambda\right|$$
$$\le 4\left(([2]_q-1)^2([4]_q-1)\right). \tag{20}$$

Upon simplifying the expressions(11) and(20), we get

$$|a_2a_4-a_3^2| \;\le\; \frac{4\cos^2\lambda}{([3]_q-1)^2}. \tag{21}$$

Choosing $c_1=c=0$ and selecting $x=-1$ in (6) and (7), we find that $c_2=-2$ and $c_3=0$. Substituting these value in (20), it is observed that equality is attained which shows that our result is sharp. This completes the proof of our Theorem 1.1. □

As $q\to 1^{-1}$ in the above Theorem we obtain the following:

Corollary 1.1. [17] *If $f(z)=z+\sum_{n=2}^{\infty}\in SPST(\lambda), (|\lambda|<\frac{\pi}{3})$ then*

$$|a_2a_4-a_3^2| \;\le\; \cos^2\lambda.$$

Remark 1.1. *If we choose $\lambda=0$, from(20), we get $|a_2a_4-a_3^2| \;\le\; \frac{4}{([3]_q-1)^2}$.*

As $q \to 1^{-1}$ in the above Remark we obtain the following:

Remark 1.2. [17] *If we choose* $\lambda = 0$, *from(20), we get* $|a_2 a_4 - a_3^2| \leq 1.$

This inequality is sharp and coincides with that of Janteng, Halim and Darus [12].

Theorem 1.2. *If* $f(z) \in CVSP(\lambda, q)$ $\left(|\lambda| \leq \frac{\pi}{2}\right)$ *then*

$$|a_2 a_4 - a_3^2| \leq \left[\frac{\{((3[3]_q[2]_q - 4[4]_q)^2 + 8[4]_q([3]_q[2]_q - [4]_q)) + T\} + L}{2[4]_q[3]^2[2]_q^2\,(2([4]_q - [3]_q) + ([3]_q[2]_q - [4]_q)\sec^2\lambda)}\right] \tag{22}$$

where $T = 4([3]_q([2]_q + 1) - 2[4]_q)^2 + 16[4]_q([4]_q - [3]_q)\cos^2\lambda$
$L = 4\left((3[3]_q[2]_q - 4[4]_q)([3]_q([2]_q + 1) - 2[4]_q)\right)\cos\lambda$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n \in CVSP(\lambda, q)$, from the Definition 1.3, there exists an analytic function $p \in \mathcal{P}$ in the unit disc E with $p(0) = 1$ and $Re\ p(z) > 0$ such that

$$\left[e^{i\lambda}\left\{1 + \frac{z\partial_q^2 f(z)}{\partial_q f(z)}\right\}\right] = p(z) \Leftrightarrow \left[e^{i\lambda}\left\{\partial_q f(z) + z\partial_q^2 f(z)\right\} - i\sin\lambda\,\partial_q f(z)\right] \tag{23}$$

$$= \cos\lambda\left\{\partial_q f(z) \times p(z)\right\}.$$

Replacing $\partial_q f(z), z\partial_q^2 f(z)$ and $p(z)$ with their equivalent series expressions in the relation (23), we have

$$\left[\left(e^{i\lambda}\left\{1 + \sum_{n=2}^{\infty}[n]_q\, a_n\, z^{n-1}\right\} + z\left\{\sum_{n=2}^{\infty}[n]_q[n-1]_q\, a_n\, z^{n-2}\right\}\right)\right.$$
$$\left. - i\sin\lambda\left\{1 + \sum_{n=2}^{\infty}[n]_q\, a_n\, z^{n-1}\right\}\right] = \left[\cos\lambda\left\{1 + \sum_{n=2}^{\infty}[n]_q\, a_n\, z^{n-1}\right\} \times \left\{1 + \sum_{n=1}^{\infty} c_n\, z^n\right\}\right].$$

Upon simplification, we obtain

$$e^{i\lambda}\left[[2]_q a_2 z + [3]_q[2]_q a_3 z^2 + [4]_q[3]_q a_4 z^3 + ...\right] = \cos\lambda\left[c_1 z + (c_2 + [2]_q c_1 a_2)z^2 \right.$$
$$\left. + (c_3 + [2]_q c_2 a_2 + [3]_q c_1 a_3)z^3 + ...\right]. \tag{24}$$

On equating the coefficients of like powers of z, z^2 and z^3 respectively in (24), after simplifying, we get

$$\left[a_2 = \frac{e^{-i\lambda}}{[2]_q} c_1 \cos\lambda, \qquad a_3 = \frac{e^{-i\lambda}}{[3]_q[2]_q}\left\{c_2 + c_1^2 e^{-i\lambda}\cos\lambda\right\}\cos\lambda\right.$$
$$\left. a_4 = \frac{e^{-i\lambda}}{[4]_q[3]_q[2]_q}\left\{[2]_q c_3 + ([2]_q + 1)\, c_1 c_2 e^{-i\lambda}\cos\lambda + c_1^3 e^{-2i\lambda}\cos^2\lambda\right\}\cos\lambda\right]. \tag{25}$$

Substituting the values of a_2, a_3 and a_4 from (25) in the second Hankel functional $|a_2 a_4 - a_3^2|$ for the function $f(z) \in CVSP(\lambda, q)$, applying the same procedure as described in Theorem 1.1, upon simplification, we obtain

$$|a_2 a_4 - a_3^2| \leq \frac{\cos^2\lambda}{[4]_q[3]_q^2[2]_q^2} \times \left|[3]_q[2]_q c_1 c_3 + ([3]_q([2]_q + 1) - 2[4]_q)\, c_1^2 c_2 \cos\lambda \right.$$
$$\left. - [4]_q c_2^2 - ([4]_q - [3]_q) c_1^4 \cos^2\lambda\right|. \tag{26}$$

Applying the same procedure as described in Theorem 1.1, after simplification, we get

$$
\begin{aligned}
&4\left|[3]_q[2]_qc_1c_3+([3]_q([2]_q+1)-2[4]_q)\,c_1^2c_2\cos\lambda-[4]_qc_2^2-([4]_q-[3]_q)c_1^4\cos^2\lambda\right|\\
&\quad\le\left|\left\{[3]_q[2]_q-[4]_q+2\left([3]_q([2]_q+1)-2[4]_q\right)\cos\lambda-4([4]_q-[3]_q)\cos^2\lambda\right\}c_1^4\right.\\
&\quad+2[3]_q[2]_qc_1(4-c_1^2)+\{2([3]_q[2]_q-[4]_q)+2([3]_q([2]_q+1)-2[4]_q)\cos\lambda\}\,c_1^2(4-c_1^2)|x|\\
&\quad\left.-\left(([3]_q[2]_q-[4]_q)c_1+4\right)(c_1+[4]_q)(4-c_1^2)|x|^2\right|.
\end{aligned}
$$

Choosing $c_1=c\in[0,2]$, using the result $(c+a)(c+b)\ge(c-a)(c-b)$, where $a,b\ge o$, applying triangle inequality an replacing $|x|$ by μ and Applying the same procedure as described in Theorem 1.1 on the right-hand side of the above inequality, we obtain

$$
\begin{aligned}
&4\left|[3]_q[2]_qc_1c_3+([3]_q([2]_q+1)-2[4]_q)\,c_1^2c_2\cos\lambda-[4]_qc_2^2-([4]_q-[3]_q)c_1^4\cos^2\lambda\right|\\
&\quad\le\left|\left\{[3]_q[2]_q-[4]_q+2\left([3]_q([2]_q+1)-2[4]_q\right)\cos\lambda-4([4]_q-[3]_q)\cos^2\lambda\right\}c^4\right.\\
&\quad+2[3]_q[2]_qc(4-c^2)+\{2([3]_q[2]_q-[4]_q)+2([3]_q([2]_q+1)-2[4]_q)\cos\lambda\}\,c^2(4-c^2)\mu\\
&\quad\left.+\left(([3]_q[2]_q-[4]_q)c-4\right)(c-[4]_q)(4-c^2)\mu^2\right|.\\
&\quad=F(c,\mu),\,with\ 0\le\mu=|x|\le1. \qquad (27)
\end{aligned}
$$

Where

$$
\begin{aligned}
F(c,\mu)=&\left|\left\{[3]_q[2]_q-[4]_q+2\left([3]_q([2]_q+1)-2[4]_q\right)\cos\lambda-4([4]_q-[3]_q)\cos^2\lambda\right\}c^4\right.\\
&+2[3]_q[2]_qc(4-c^2)+\{2([3]_q[2]_q-[4]_q)+2([3]_q([2]_q+1)-2[4]_q)\cos\lambda\}\,c^2(4-c^2)\mu\\
&\left.+\left(([3]_q[2]_q-[4]_q)c-4\right)(c-[4]_q)(4-c^2)\mu^2\right|.
\end{aligned}
$$

Applying the same procedure as described in Theorem1.1, we get

$$
\frac{\partial F}{\partial\mu}=\left[\left(2([3]_q[2]_q-[4]_q)+2([3]_q[2]_q-2[4]_q)\cos\lambda\right)c^2+2\mu\left(([3]_q[2]_q-[4]_q)c-4\right)(c-[4]_q)\right]\times(4-c^2). \qquad (28)
$$

From (28), for $0<\mu<1$, c with $0<c<2$ and $0<q<1$ for a fixed $\lambda(|\lambda|\le\frac{\pi}{2})$, we observe that $\frac{\partial F}{\partial\mu}>0$. Consequently, $F(c,\mu)$ is an increasing function of μ and hence cannot have a maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$. Further, for fixed $c\in[0,2]$, we have

$$
\max_{0\le\mu\le1}F(c,\mu)=F(c,1)=G(c). \qquad (29)
$$

In view of the expression(29), replacing μ by 1 in (27), upon simplification, we obtain

$$
\begin{aligned}
G(c)=&\left\{-2\left(([3]_q[2]_q-[4]_q)+2([4]_q-[3]_q)\cos^2\lambda\right)c^4\right.\\
&\left.+4\left((3[3]_q[2]_q-4[4]_q)+2([3]_q([2]_q+1)-2[4]_q)\cos\lambda\right)c^2+16[4]_q\right\} \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
G'(c)=&\left\{-8\left(([3]_q[2]_q-[4]_q)+2([4]_q-[3]_q)\cos^2\lambda\right)c^3\right.\\
&\left.+8\left((3[3]_q[2]_q-4[4]_q)+2([3]_q([2]_q+1)-2[4]_q)\cos\lambda\right)c\right\} \qquad (31)
\end{aligned}
$$

$$G''(c) = \left\{-24\left(([3]_q[2]_q - [4]_q) + 2([4]_q - [3]_q)\cos^2\lambda\right)c^2 \right.$$
$$\left. +8\left((3[3]_q[2]_q - 4[4]_q) + 2([3]_q([2]_q+1) - 2[4]_q)\cos\lambda\right)\right\} \quad (32)$$

To obtain optimum value of $G(c)$, consider $G'(c) = 0$. From(31), we get

$$-8c\left\{\left(([3]_q[2]_q - [4]_q) + 2([4]_q - [3]_q)\cos^2\lambda\right)c^2 \right.$$
$$\left. -\left((3[3]_q[2]_q - 4[4]_q) + 2([3]_q([2]_q+1) - 2[4]_q)\cos\lambda\right)\right\} = 0 \quad (33)$$

Let us discuss the following cases:
Case 1 :If $c = 0$, then from (32) we obtain

$$8\left((3[3]_q[2]_q - 4[4]_q) + 2([3]_q([2]_q+1) - 2[4]_q)\cos\lambda\right) > 0,\ for|\lambda| \le \frac{\pi}{2}$$

Therefore, by the second derivative test, $G(c)$ has minimum value at $c = 0$.
case 2 : If $c \neq 0$, then from (33) we get

$$c^2 = \left\{\frac{((3[3]_q[2]_q - 4[4]_q) + 2([3]_q([2]_q+1) - 2[4]_q)\cos\lambda)}{(([3]_q[2]_q - [4]_q) + 2([4]_q - [3]_q)\cos^2\lambda)}\right\}, \quad (34)$$

Using the value of c^2 in (32), after simplifying, we get

$$G''(c) = -16\left\{((3[3]_q[2]_q - 4[4]_q) + 2([3]_q([2]_q+1) - 2[4]_q)\cos\lambda)\right\} \quad (35)$$

by the second derivative test, $G(c)$ has maximum value at c, where c^2 is given in (34). using the value of c^2 in (30), upon simplification, we obtain

$$\max_{0\le c\le 2} G(c) = 2\left[\frac{\{((3[3]_q[2]_q - 4[4]_q)^2 + 8[4]_q([3]_q[2]_q - [4]_q)) + T\} + L}{(([3]_q[2]_q - [4]_q) + 2([4]_q - [3]_q)\cos^2\lambda)}\right] \quad (36)$$

where $T = 4([3]_q([2]_q+1) - 2[4]_q)^2 + 16[4]_q([4]_q - [3]_q)\cos^2\lambda$
$L = 4\left((3[3]_q[2]_q - 4[4]_q)([3]_q([2]_q+1) - 2[4]_q)\right)\cos\lambda$
Considering, the maximum value of $G(c)$ at c, where c^2 is given in (34), from(27) and(36), after simplifying, we get

$$\left|[3]_q[2]_q c_1 c_3 + ([3]_q([2]_q+1) - 2[4]_q)\,c_1^2 c_2\cos\lambda - [4]_q c_2^2 - ([4]_q - [3]_q)c_1^4\cos^2\lambda\right| \le M \quad (37)$$

where $M = \left[\frac{\{\left((3[3]_q[2]_q - 4[4]_q)^2 + 8[4]_q([3]_q[2]_q - [4]_q)\right) + T\} + L}{2(([3]_q[2]_q - [4]_q) + 2([4]_q - [3]_q)\cos^2\lambda)}\right]$
and $T = 4([3]_q([2]_q+1) - 2[4]_q)^2 + 16[4]_q([4]_q - [3]_q)\cos^2\lambda$
$L = 4\left((3[3]_q[2]_q - 4[4]_q)([3]_q([2]_q+1) - 2[4]_q)\right)\cos\lambda$
From the expressions(26) and (37), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \le \left[\frac{\{((3[3]_q[2]_q - 4[4]_q)^2 + 8[4]_q([3]_q[2]_q - [4]_q)) + T\} + L}{2[4]_q[3]^2[2]_q^2\left(2([4]_q - [3]_q) + ([3]_q[2]_q - [4]_q)\sec^2\lambda\right)}\right] \quad (38)$$

where $T = 4([3]_q([2]_q+1) - 2[4]_q)^2 + 16[4]_q([4]_q - [3]_q)\cos^2\lambda$
$L = 4\left((3[3]_q[2]_q - 4[4]_q)([3]_q([2]_q+1) - 2[4]_q)\right)\cos\lambda$
This completes the proof of our Theorem. □

As $q \to 1^{-1}$ in the above Theorem we obtain the following:

Corollary 1.2. [17]*If* $f(z) \in CVSP(\lambda)$ $\left(|\lambda| \leq \frac{\pi}{2}\right)$ *then*

$$|a_2a_4 - a_3^2| \leq \left[\frac{17(1+\cos^2\lambda) + 2\cos\lambda}{144(1+\sec^2\lambda)}\right]$$

Remark 1.3. *If we choose* $\lambda = 0$*, from(38), we get*

$$|a_2a_4 - a_3^2| \leq \left[\frac{\{((3[3]_q[2]_q - 4[4]_q)^2 + 8[4]_q([3]_q[2]_q - [4]_q)) + A\} + B}{2[4]_q[3]^2[2]_q^2\,(2([4]_q - [3]_q) + ([3]_q[2]_q - [4]_q))}\right],$$

where $A = 4([3]_q([2]_q + 1) - 2[4]_q)^2 + 16[4]_q([4]_q - [3]_q)$
$B = 4\left((3[3]_q[2]_q - 4[4]_q)([3]_q([2]_q + 1) - 2[4]_q)\right)$

As $q \to 1^{-1}$ in the above Remark we obtain the following:

Remark 1.4. [17] *If we choose* $\lambda = 0$*, from(38), we get* $|a_2a_4 - a_3^2| \leq \frac{1}{8}$.

This inequality is sharp and coincides with that of Janteng, Halim and Darus []

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