q-ANALOGUE OF JANOWSKI FUNCTIONS WITH NEGATIVE COEFFICIENTS

HAMID SHAMSAN[1], S. LATHA[2]

[1,2]Department of Mathematics, Yuvaraja's College, University of Mysore,

Mysore 570 005, INDIA

Abstract. In this paper, the concept of q-derivative, Janowski functions with negative coefficients are combined to define $\mathcal{ST}(A, B, q)$. We derive a necessary and sufficient condition, distortion theorem, and neighborhood results. We also establish extreme point results, some results concerning the partial sums for the function $f(z)$ belonging to the class $\mathcal{ST}(A, B, q)$.

Keywords: Janowski functions, Subordination, q-Derivative, Neighborhoods, Partial sums, Extreme points.

2010 AMS Subject Classification: 30C45.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}$, and suppose $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions that are univalent in $\mathcal{U}$.

In this note we give characterizations for q-analogue classes related to the Janowski class in terms of negative coefficients. The intrinsic properties of q-analogues including the applications in the study of quantum groups and q-deformed subalgebras and study the fractals are known in the literature. Some integral transforms in the classical analysis have their q-analogues in the theory of q-calculus. This has led various researcher in the field of q-theory for extending important results in classical analysis to their q-analogues.

We recall the following neighborhood concept introduced by Goodman [2] and generalized by Ruscheweyh [5]

E-mail addresses: hmas19771@gmail.com (author 1), drlatha@gmail.com (author 2).

Definition 1.1. *For any $f \in \mathcal{A}$, r-neighborhood of function f can be defined as:*

$$\mathcal{N}_r(f) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n|a_n - b_n| \leq r \right\}. \tag{1.2}$$

For $e(z) = z$, we can see that

$$\mathcal{N}_r(e) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n|b_n| \leq r \right\}. \tag{1.3}$$

Also, let Ω be the family of functions w, analytic in $\mathcal{U}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$. If f and g are analytic in $\mathcal{U}$, we say that a function f is subordinate to a function g in $\mathcal{U}$, if there exists a function $w \in \Omega$ such that $f(z) = g(w(z))$, and we denote this by $f \prec g$. If g is univalent in $\mathcal{U}$ then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part.

Definition 1.2. [1] *Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\operatorname{Re} p(z) > 0$, $z \in \mathcal{U}$.*

Any function p in $\mathcal{P}$ has the representation $p(z) = \dfrac{1 + w(z)}{1 - w(z)}$, where $w \in \Omega$ and

$$\Omega = \{ w \in \mathcal{A} : w(0) = 0, |w(z)| < 1 \}. \tag{1.4}$$

Definition 1.3. [4] *Let $\mathcal{P}[A, B]$, with $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \dfrac{1 + Aw(z)}{1 + Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$.*

We observe that $p \in \mathcal{P}[A, B]$ if and only if $p(z) \prec \dfrac{1 + Az}{1 + Bz}$. Jackson[3] initiated q-calculus and developed the concept of the q-integral and q-derivative.

For a function $f \in \mathcal{S}$ given by (1.1) and $0 < q < 1$, the q-derivative of f is defined by

Definition 1.4.

$$\partial_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0, \qquad 0 < q < 1. \end{cases} \tag{1.5}$$

Equivalently (1.5), may be written as $\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$ where $[n]_q = \frac{1-q^n}{1-q}$. Note that as $q \to 1$, $[n]_q \to n$.

Definition 1.5. *A function $f \in \mathcal{A}$ is said to belongs to the class $\mathcal{S}_q(A,B)$, with $-1 \le B < A \le 1$ and $0 < q < 1$ if*

$$\frac{z\partial_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz}.$$

We denote by $\mathcal{T}$ for the class of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \ge 0), \tag{1.6}$$

that are analytic in the open unit disk $\mathcal{U}$. We denote by $\mathcal{ST}(A,B,q)$, the class obtained by taking the intersection of $\mathcal{S}_q(A,B)$ with $\mathcal{T}$,

$$\mathcal{ST}(A,B,q) = \mathcal{S}_q(A,B) \cap \mathcal{T}.$$

In this paper, we derive a necessary and sufficient condition, distortion theorem, partial sums and neighborhood result for this new class.

2. Main results

Theorem 2.1. *Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \ge 0)$, be analytic function in $\mathcal{U}$, then $f \in \mathcal{ST}(A,B,q)$ if and only if*

$$\sum_{n=2}^{\infty} \{[n]_q - 1) + |B[n]_q - A|\} a_n \le (A-B) \tag{2.1}$$

for $-1 \le B < A \le 1$, $0 < q < 1$ and $B \ge \frac{A}{[2]_q}$. The result is sharp.

Proof. Assume the inequality (2.1) holds and let $|z| = 1$, it needs to show that the values satisfy the condition

$$\left| \frac{z\partial_q f(z) - f(z)}{Af(z) - Bz\partial_q f(z)} \right| \le 1, \tag{2.2}$$

We have

$$\left| \frac{z\partial_q f(z) - f(z)}{Af(z) - Bz\partial_q f(z)} \right|$$

$$= \left| \frac{\sum_{n=2}^{\infty}([n]_q - 1)a_n z^{n-1}}{(A-B) - \sum_{n=2}^{\infty}(B[n]_q - A)a_n z^{n-1}} \right|$$

$$\le \frac{\sum_{n=2}^{\infty}([n]_q - 1)a_n |z|^{n-1}}{(A-B) - \sum_{n=2}^{\infty}|B[n]_q - A| a_n |z|^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty}([n]_q - 1)a_n}{(A-B) - \sum_{n=2}^{\infty}|B[n]_q - A|a_n}.$$

This last expression is bounded above by 1, which implies that $f \in \mathcal{ST}(A, B, q)$. Conversely, assume that $f \in \mathcal{ST}(A, B, q)$, then

$$\Re\left\{\frac{z\partial_q f(z) - f(z)}{Af(z) - Bz\partial_q f(z)}\right\} \leq \left|\frac{z\partial_q f(z) - f(z)}{Af(z) - Bz\partial_q f(z)}\right| \leq 1,$$

or

$$\Re\left\{\frac{z\partial_q f(z) - f(z)}{Af(z) - Bz\partial_q f(z)}\right\} \leq 1, \tag{2.3}$$

$$\Re\left\{\frac{\sum_{n=2}^{\infty}([n]_q - 1)a_n z^{n-1}}{(A-B) - \sum_{n=2}^{\infty}(B[n]_q - A)a_n z^{n-1}}\right\} \leq 1,$$

Choose values of z on the real axis so that $\frac{z\partial_q f(z) - f(z)}{Af(z) - Bz\partial_q f(z)}$, is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=2}^{\infty}([n]_q - 1)a_n \leq (A-B) - \sum_{n=2}^{\infty}|B[n]_q - A|a_n,$$

which gives (2.1). The coefficient inequality (2.1) is sharp for the analytic function

$$g(z) = z - \frac{A-B}{([n]_q - 1) + |B[n]_q - A|}z^n,$$

where $-1 \leq B < A, B \geq \frac{A}{[2]_q}$ and $0 < q < 1$. □

Theorem 2.2. *Let $f \in \mathcal{ST}(A, B, q)$, then*

$$|z| - \sum_{n=2}^{i} a_n|z|^n - \tau_i|z|^{i+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{i} a_n|z|^n + \tau_i|z|^{i+1},$$

where

$$\tau_i = \frac{(A-B) - \sum_{n=2}^{i}([n]_q - 1) + |B[n]_q - A|a_n}{(i+1)}.$$

Proof. From Theorem 2.1 we have

$$\sum_{n=i+1}^{\infty}[([n]_q - 1) + |B[n]_q - A|]a_n$$

$$\leq (A-B) - \sum_{n=2}^{i}[[n]_q - 1) + |B[n]_q - A|]a_n.$$

On the other hand

$$([n]_q - 1) + |B[n]_q - A| \geq ([n]_q - 1) \tag{2.4}$$

and hence $([n]_q - 1)$ is monotonically increasing with respect to n. So we can write

$$(i+1)\sum_{n=i+1}^{\infty} a_n \leq (A-B) - \sum_{n=2}^{i}[([n]_q - 1) + |B[n]_q - A|]a_n,$$

which implies that

$$\sum_{n=i+1}^{\infty} a_n \leq \tau_i,$$

hence we have

$$|f(z)| \leq |z| + \sum_{n=2}^{i} |a_n||z|^n + \tau_i|z|^{i+1},$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{i} a_n|z|^n - \tau_i|z|^{i+1}.$$

This completes the proof of theorem. □

Theorem 2.3.

$$\mathcal{ST}(A, B, q) \subseteq \mathcal{N}_\rho(e),$$

where

$$\rho = \frac{3(A-B)}{2}.$$

and $-1 \leq B < A \leq 1, B \geq \frac{A}{[2]_q}, b > 0, \sigma > -1, 0 < q < 1.$

Proof. For $f \in \mathcal{ST}(A, B, q)$, (2.4) yields

$$2\sum_{n=2}^{\infty} a_n \leq (A-B),$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{(A-B)}{2}. \tag{2.5}$$

On the other hand, from Theorem 2.1, we have,

$$\sum_{n=2}^{\infty}([n]_q - 1)a_n \leq (A-B).$$

Equivalently

$$\sum_{n=2}^{\infty}[n]_q a_n \leq (A-B) + \sum_{n=2}^{\infty} a_n,$$

that is,

$$\sum_{n=2}^{\infty}[n]_q a_n \leq \frac{3(A-B)}{2} = \rho,$$

which, in view of the Definition 1.1, proves Theorem 2.3. □

Corollary 2.4. *Let* $f \in \mathcal{ST}(A,B,q)$, *then*

$$a_n \leq \frac{A-B}{([n]_q-1)+|B[n]_q-A|}, \qquad n \geq 2. \tag{2.6}$$

Now we derive certain results about extreme points of the class $\mathcal{ST}(A,B,q)$

Theorem 2.5. *Let* $f_1(z) = z$, *and*

$$f_n(z) = z - \frac{A-B}{([n]_q-1)+|B[n]_q-A|} z^n \tag{2.7}$$

then,

$f(z) \in \mathcal{ST}(A,B,q)$ *if and only if it be expressed in the following form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \qquad \lambda_n \geq 0, \ \sum_{n=1}^{\infty} \lambda_n = 1. \tag{2.8}$$

Proof. Suppose that

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\
&= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z). \\
&= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \left(z - \frac{A-B}{([n]_q-1)+|B[n]_q-A|} z^n \right) \\
&= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n z - \sum_{n=2}^{\infty} \lambda_n \frac{A-B}{([n]_q-1)+|B[n]_q-A|} z^n \\
&= \left(\sum_{n=1}^{\infty} \lambda_n \right) z - \sum_{n=2}^{\infty} \lambda_n \frac{A-B}{([n]_q-1)+|B[n]_q-A|} z^n \\
&= z - \sum_{n=2}^{\infty} \lambda_n \frac{A-B}{([n]_q-1)+|B[n]_q-A|} z^n.
\end{aligned}$$

Then

$$\sum_{n=2}^{\infty} \lambda_n \left(\frac{A-B}{([n]_q-1)+|B[n]_q-A|} \right) \left(\frac{([n]_q-1)+|B[n]_q-A|}{A-B} \right) = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

Thus by Theorem 2.1, we get $f \in \mathcal{ST}(A, B, q)$.

Conversely suppose that $f \in \mathcal{ST}(A, B, q)$, by Corollary 2.4, we have

$$a_n \leq \frac{A - B}{([n]_q - 1) + |B[n]_q - A|}, \qquad n \geq 2,$$

setting

$$\lambda_n = \frac{A - B}{([n]_q - 1) + |B[n]_q - A|} a_n, \qquad \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

We have

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

which completes the proof. □

Theorem 2.6. *Let the functions* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0$ *and* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0$ *be in the class* $\mathcal{ST}(A, B, q)$. *Then for* $0 \leq \zeta \leq 1$,

$$h(z) = (1 - \zeta) f(z) + \zeta g(z) = z - \sum_{n=2}^{\infty} c_n z^n, (c_n \geq 0),$$

is in the class $\mathcal{ST}(A, B, q)$.

Proof. Suppose that $f(z), g(z) \in \mathcal{ST}(A, B, q)$. From Theorem 2.1 we have

$$\sum_{n=2}^{\infty} \{([n]_q - 1) + |B[n]_q - A|\} a_n \leq (A - B),$$

and

$$\sum_{n=2}^{\infty} \{([n]_q - 1) + |B[n]_q - A|\} b_n \leq (A - B).$$

We can see that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \{([n]_q - 1) + |B[n]_q - A|\} c_n \\
&= \sum_{n=2}^{\infty} \{([n]_q - 1) + |B[n]_q - A|\} [(1 - \zeta) a_n + \zeta b_n] \\
&= (1 - \zeta) \left\{ \sum_{n=2}^{\infty} \{([n]_q - 1) + |B[n]_q - A|\} a_n \right\} \\
&\qquad + \zeta \left\{ \sum_{n=2}^{\infty} \{([n]_q - 1) + |B[n]_q - A|\} b_n \right\} \\
&\leq (1 - \zeta)(A - B) + \zeta (A - B) = (A - B),
\end{aligned}$$

which completes the proof. □

3. PARTIAL SUMS

In this section we will examine the ratio of a function of the form (1.6) to its sequence of partial sums defined by $f_1(z) = z$ and $f_k(z) = z - \sum_{n=2}^{k} a_n z^n$ when the coefficients of f are sufficiently small to satisfy the condition (2.1). We will determine sharp lower bounds for $\Re\left\{\frac{f(z)}{f_k(z)}\right\}$,$\Re\left\{\frac{f_k(z)}{f(z)}\right\}$,$\Re\left\{\frac{\partial_q f(z)}{\partial_q f_k(z)}\right\}$ and $\Re\left\{\frac{\partial_q f_k(z)}{\partial_q f(z)}\right\}$

Theorem 3.1. *If $f \in \mathcal{ST}(A, B, q)$, then*

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq 1 - \frac{1}{c_{k+1}}, \quad (z \in \mathcal{U}, k \in \mathbb{N}), \tag{3.1}$$

and

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{c_{k+1}}{1 + c_{k+1}}, \quad (z \in \mathcal{U}, k \in \mathbb{N}), \tag{3.2}$$

where $c_k = \frac{([k]_q - 1) + |B[k]_q - A|}{A - B}$. The estimates in (3.1) and (3.2) are sharp.

Proof. Suppose that $f \in \mathcal{ST}^{(j,k)}(A, B)$, by Theorem 2.1, we have

$$f \in \mathcal{ST}^{(j,k)}(A, B) \Leftrightarrow \sum_{n=2}^{\infty} c_n a_n \leq 1,$$

It is easy to verify that $c_{n+1} > c_n > 1$. Thus,

$$\sum_{n=2}^{k} a_n + c_{k+1} \sum_{n=k+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} c_n a_n \leq 1. \tag{3.3}$$

We may write

$$c_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{c_{k+1}}\right)\right\} = \frac{1 - \sum_{n=2}^{\infty} a_n z^{n-1} - c_{k+1} \sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} = \frac{1 + D(z)}{1 + E(z)}.$$

Set

$$\frac{1 + D(z)}{1 + E(z)} = \frac{1 - w(z)}{1 + w(z)},$$

so that

$$w(z) = \frac{E(z) - D(z)}{2 + E(z) + D(z)},$$

then

$$w(z) = \frac{c_{k+1} \sum_{n=k+1}^{\infty} a_n z^{n-1}}{2 - 2\sum_{n=2}^{k} a_n z^{n-1} - c_{k+1} \sum_{n=k+1}^{\infty} a_n z^{n-1}},$$

and

$$|w(z)| \leq \frac{c_{k+1} \sum_{n=k+1}^{\infty} a_n}{2 - 2\sum_{n=2}^{k} a_n - c_{k+1} \sum_{n=k+1}^{\infty} a_n}.$$

INTERNATIONAL JOURNAL OF INFORMATION AND COMPUTING SCIENCE ISSN NO: 0972-1347

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{k} a_n + c_{k+1} \sum_{n=k+1}^{\infty} a_n \leq 1,$$

which is true by (3.3). This readily yields the assertion (3.1).

To see that

$$f(z) = z - \frac{z^{k+1}}{c_{k+1}}, \tag{3.4}$$

gives sharp results, we observe that

$$\frac{f(z)}{f_k(z)} = 1 - \frac{z^k}{c_{k+1}}.$$

Letting $z \to 1^-$, we have

$$\frac{f(z)}{f_k(z)} = 1 - \frac{1}{c_{k+1}},$$

which shows that the bounds in (3.1) are the best possible for each $n \in \mathbb{N}$.

In the same way we take

$$(1 + c_{k+1}) \left(\frac{f_k(z)}{f(z)} - \frac{c_{k+1}}{1 + c_{k+1}} \right) = \frac{1 - \sum_{n=2}^{\infty} a_n z^{n-1} + c_{m+1} \sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} = \frac{1 - w(z)}{1 + w(z)},$$

where

$$|w(z)| \leq \frac{1 + c_{k+1} \sum_{n=m+1}^{\infty} a_n}{2 - 2\sum_{n=2}^{\infty} a_n + (1 - c_{m+1}) \sum_{n=m+1}^{\infty} a_n}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{k} a_n + c_{k+1} \sum_{n=k+1}^{\infty} a_n \leq 1,$$

which is true by (3.3). This readily yields the assertion (3.2).

The estimate in (3.2) is sharp with the extremal function $f(z)$ given by (3.4). This completes the proof of Theorem. □

Theorem 3.2. *If f If f of the form* (1.1) *and satisfies condition* (2.1), *then*

$$\Re \left\{ \frac{\partial_q f(z)}{\partial_q f_k(z)} \right\} \geq 1 - \frac{[k]_q + 1}{c_{k+1}}, \quad (z \in \mathcal{U}, k \in \mathbb{N}), \tag{3.5}$$

and

$$\Re \left\{ \frac{\partial_q f_k(z)}{\partial_q f(z)} \right\} \geq \frac{c_{k+1}}{[k]_q + 1 + c_{k+1}}, \quad (z \in \mathcal{U}, k \in \mathbb{N}). \tag{3.6}$$

where $c_k = \frac{([k]_q - 1) + |B[k]_q - A|}{A - B}$. *The estimates in* (3.5) *and* (3.6) *are sharp with the extremal function given by* (3.4).

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