FEKETE-SZEGO INEQUALITIES FOR FUNCTIONS WITH RESPECT TO (j,k)-SYMMETRICAL FUNCTIONS

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Abstract. In this paper sharp upper bounds of $|a_3 - \mu a_2^2|$ for functions belonging to new subclasses defined using the concept of (j,k)-symmetric functions are derived. Certain applications for functions defined through fractional derivatives in the sense of Riemann Liouville are discussed .

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1. Introduction

Let $\mathscr{A}$ denote the class of functions of form

(1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathscr{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$, and $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of all function which are univalent in $\mathscr{U}$.

In 1916 Bieberbach [15], proved his famous conjecture. The Bieberbach conjecture: The coefficients of each function $f \in \mathscr{S}$ satisfy $|a_2| \leq 2$. In 1923 Löwner [14] proved that $|a_3| \leq 3$ for function $f \in \mathscr{S}$, With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and a_2^2 for the class $\mathscr{S}$, Fekete and Szegö [16] used Löwners method to

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prove the following well known result for the class $\mathcal{S}$, for real number μ and $a_1 = 1$.

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu, & \mu \leq 0; \\ 1+2\exp(\frac{-2\mu}{1-\mu}), & 0 \leq \mu \leq 1; \\ 4\mu-3, & \mu \geq 1. \end{cases}$$

The above inequality plays a very important role in determining estimates of higher coefficients for some subclasses of $\mathcal{S}$.

Definition 1.1. *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0,1,2,..,k-1$ where $k = 2,3,...$ is a natural number. A function $f : \mathcal{U} \mapsto \mathbb{C}$ is called (j,k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z),\ z \in \mathcal{U}.$$

The family of all (j,k)-symmetrical functions is denoted be $\mathcal{S}^{(j,k)}$. $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are respectively the classes of even, odd and k-symmetric functions. We have the following decomposition theorem.

Theorem 1.2. [12] *For every mapping f is k-fold symmetric function, there exists exactly a unique sequence of (j,k)- symmetrical functions $f_{j,k}$,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj}\left(\sum_{n=1}^{\infty} a_n(\varepsilon^v z)^n\right),$$

then

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \psi_n a_n z^n,\ a_1 = 1 \qquad \psi_n = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(n-j)v}, \tag{2}$$

where

$$(f \in \mathcal{A};\ k = 1,2,...;\ j = 0,1,2,...,k-1).$$

The notion of (j,k)-symmetrical functions $(k = 2,3,\dots ; j = 0,1,2,\dots,k-1)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [12].

In [8] Fuad Al-sarari and Latha studied Fekete-Szego Inequalities and (j,k)-symmetric Functions, to know more about the classes with respect to (j,k)symmetic points see [7, 9, 10, 11].

Definition 1.3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots.$ be univalent starlike with respect to* 1 *which maps the unit disk $\mathscr{U}$ onto a region in the right half plane which is symmetric with respect to the real axis. Let $0 \le \beta \le \alpha \le 1$ and $B_1 > 0$. Then the function $f(z) \in \mathscr{A}$ is in the class $\mathscr{K}^{j,k}(\alpha,\beta,\phi)$ if*

$$\frac{\alpha\beta z^3 f'''(z) + (2\alpha\beta + \alpha - \beta) z^2 f''(z) + z f'(z)}{\alpha\beta z^2 f''_{j,k}(z) + (\alpha - \beta) z f'_{j,k}(z) + (1 - \alpha + \beta) f_{j,k}(z)} \prec \phi(z).$$

We note that for suitable choices j,k,α,β,ϕ we obtain the following classes in literature ;

(i)$\mathscr{K}^{j,k}(0,0,\phi) = S_s^{j,k}(0)$ the class introduced by Fuad Alsarari and Latha [8].

(ii)$\mathscr{K}^{1,k}(0,0,\phi) = S_s^{k}(\phi)$ the class introduced by Al-Shaqsi and Darus in [1].

(iii)$\mathscr{K}^{1,k}(0,0,\frac{1+Az}{1+Bz}) = S_s^{k}[A,B]$ we get the class introduced by Al-Shaqsi and Darus in [1].

(iv)$\mathscr{K}^{1,2}(0,0,\Phi) = S_s^{*}(\Phi)$ the class introduced by Shanmugam et al. in [13].

(v)$\mathscr{K}^{1,k}(\alpha,0,\phi)$ we get the class introduced by Parvatham and Radha [4].

(vi)$\mathscr{K}^{1,1}(0,0,\phi) = \mathscr{S}^{*}(\phi)$ we have the class studied by see Ma and Minda [2].

2. **Fekete-Szegö Inequality**

To prove our results, we need the following lemmas.

Lemma 2.1. [2] *If* $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is analytic function with positive real part in* $\mathscr{U}$ *and* v *is complex number, then*

$$|c_2 - vc_1^2| \leq 2\max\{1, |2v-1|\},$$

the result is sharp for functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = \frac{1+z}{1-z}.$$

Lemma 2.2. [2] *If* $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is analytic function with positive real part in* $\mathscr{U}$*, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2, & v \leq 0; \\ 2, & 0 \leq v \leq 1; \cdot \\ 4v-2, & v \geq 1. \end{cases} \tag{3}$$

When $v < 0$ *or* $v > 1$ *the equality holds if and only if* $p(z) = (1+z)/(1-z)$ *or one of its rotations. If* $0 < v < 1$*, then the equality if and only if* $p(z) = (1+z^2)/(1-z^2)$ *or one of its rotations. If* $v = 0$ *the equality holds if and only if*

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\vartheta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\vartheta\right)\frac{1-z}{1+z}, \ (0 \leq \vartheta \leq 1),$$

or one of its rotations. If $v = 1$*, the equality holds if and only if*

$$\frac{1}{p(z)} = \left(\frac{1}{2} + \frac{1}{2}\vartheta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\vartheta\right)\frac{1-z}{1+z}, \ (0 \leq \vartheta \leq 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0 < v < 1$*:*

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2, \qquad (0 < v < 1/2),$$

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2, \qquad (1/2 < v < 1),$$

Theorem 2.3. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots (B_1 > 0)$*. If* $f(z)$ *given by* (1) *belongs to* $\mathscr{K}^{j,k}(\alpha, \beta, \phi)$*, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))}\left[B_2 + \frac{B_1^2\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))} - \mu\frac{B_1^2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right] & \mu \leq \sigma_1; \\ \frac{B_1}{(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}, & \sigma_1 \leq \mu \leq \sigma_2;, \\ -\frac{1}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))}\left[B_2 + \frac{B_1^2\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))} - \mu\frac{B_1^2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right], & \mu \geq \sigma_2. \end{cases}$$

where

$$\sigma_1 = \frac{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))^2}{B_1(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))}\left[-1 + \frac{B_2}{B_1} + \frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))}\right],$$

and

$$\sigma_2 = \frac{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))^2}{B_1(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))}\left[1 + \frac{B_2}{B_1} + \frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))}\right],$$

where $\psi_n, \delta_n(\alpha,\beta)$ *and* $\chi_n(\alpha,\beta)$ *are defined respectively as*

(4) $$\delta_n(\alpha,\beta) = n[(n-1)(n-2)\alpha\beta + (n-1)(2\alpha\beta + \alpha - \beta) + 1],$$

and

(5) $$\chi_n(\alpha,\beta) = n(n-1)\alpha\beta + n(\alpha - \beta) + 1 - \alpha + \beta,$$

.

The result is sharp.

Proof. Let $f(z) \in \mathscr{K}^{j,k}(\alpha,\beta,\phi)$, then there is a Schwarz function $w(z)$ in $\mathscr{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathscr{U}$ such that

$$\frac{\alpha\beta z^3 f'''(z) + (2\alpha\beta + \alpha - \beta)z^2 f''(z) + zf'(z)}{\alpha\beta z^2 f''_{j,k}(z) + (\alpha - \beta)zf'_{j,k}(z) + (1 - \alpha + \beta)f_{j,k}(z)} = \phi(w(z)).$$

If $p_1(z)$ is analytic and has positive real part in $\mathscr{U}$ and $p_1(0) = 1$ then

(6) $$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + ..., \ z \in \mathscr{U},$$

from (6), we have

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + ...,$$

therefore, we have

(7) $$p(z) = \phi(w(z)) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + ...$$

ψ_n is defined by 2 and

Now let

$$p(z) = \frac{\alpha\beta z^3 f'''(z) + (2\alpha\beta + \alpha - \beta)z^3 f'(z))' + zf'(z)}{\alpha\beta z^2 f''_{j,k}(z) + (\alpha - \beta)zf'_{j,k}(z) + (1 - \alpha + \beta)f_{j,k}(z)} = 1 + d_1 z + d_2 z^2 + ...,$$

or

$$\frac{1+\sum_{n=2}^{\infty}\delta_n(\alpha,\beta)a_nz^{n-1}}{\sum_{n=1}^{\infty}\psi_n\chi_n(\alpha,\beta)a_nz^{n-1}} = 1+d_1z+d_2z^2+...,\tag{8}$$

Equating coefficients of z^n on both sides, we have

$$d_1 = a_2(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta)),\tag{9}$$

and

$$d_2 = a_3(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta)) - a_2^2(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))\psi_2\chi_2(\alpha,\beta).\tag{10}$$

We note that $\delta_1(\alpha,\beta)=\chi_1(\alpha,\beta)=\psi_1=a_1=1, \qquad and \qquad (\delta_n(\alpha,\beta)-\chi_n(\alpha,\beta))\geq 1.$

Now from (7),(8),(9) and (10) we get

$$a_2 = \frac{B_1c_1}{2(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))},$$

and

$$a_3 = \frac{B_1}{2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[c_2-\frac{c_1^2}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}\right]\right].$$

Therefore, we have

$$a_3-\mu a_2^2 = \frac{B_1}{2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}[c_2-vc_1^2],$$

where

$$v = \frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}+\mu\frac{B_1(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right].$$

Our result now follows by an application of Lemma 2.2.

To show that these bounds are sharp, we define the functions $K_{\phi_m},\ m=2,3,...$ by

$$\frac{\alpha\beta z^3K_{\phi_m}'''(z)+(2\alpha\beta+\alpha-\beta)z^2K_{\phi_m}''(z)+zK_{\phi_m}'(z)}{\alpha\beta z^2K_{\phi_m(j,k)}''(z)+(\alpha-\beta)zK_{\phi_m(j,k)}'(z)+(1-\alpha+\beta)K_{\phi_m(j,k)}(z)} = \phi(z^{n-1}).$$

$$K_{\phi_m}(0)=0=[K_{\phi_m}]'(0)-1$$

and the function F_λ and G_λ $(0\leq\lambda\leq 1)$

$$\frac{\alpha\beta z^3F_\lambda'''(z)+(2\alpha\beta+\alpha-\beta)z^2F_\lambda''(z)+zF_\lambda'(z)}{\alpha\beta z^2F_{\lambda(j,k)}''(z)+(\alpha-\beta)zF_{\lambda(j,k)}'(z)+(1-\alpha+\beta)F_{\lambda(j,k)}(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right).$$

$$F_\lambda(0) = 0 = F'_\lambda(0) - 1,$$

and

$$\frac{\alpha\beta z^3 G'''_\lambda(z) + (2\alpha\beta + \alpha - \beta) z^2 G''_\lambda(z) + zG'_\lambda(z)}{\alpha\beta z^2 G''_{\lambda(j,k)}(z) + (\alpha - \beta) zG'_{\lambda(j,k)}(z) + (1 - \alpha + \beta) G_{\lambda(j,k)}(z)} = \phi\left(\frac{-z(z+\lambda)}{1+\lambda z}\right).$$

$$G_\lambda(0) = 0 = G'_\lambda(0) - 1.$$

Obviously the functions $K_{\phi_m}, F_\lambda, G_\lambda \in \mathscr{K}^{j,k}(\alpha,\beta,\phi)$. Also we write $K_\phi := K_{\phi_2}$ if $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is K_{ϕ_3} or one of its rotations. $\mu = \sigma_1$ then equality holds if and only if f is F_λ or one of its rotations. $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations. If $\sigma_1 \le \mu \le \sigma_2$, in view of Lemma 2.2. this completes the proof of Theorem . □

Remark 2.4. *(i)For $\alpha = \beta = 0, j = 1$ we get the result obtained by by Al-Shaqsi and Darus* [1].
(ii)For $\alpha = \beta = 0, j = 1, k = 2$ we obtain the result obtained by Shanmugam et al. in [13].
(iii)For $\alpha = \beta = 0, j = 1, k = 1$ in Theorem 2.3 we arrive to the result obtained by Ma and Minda [2].

If $\sigma_1 \le \mu \le \sigma_2$ in view Lemma 2.2 and Theorem 2.3 can be improved.

Theorem 2.5. $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots (B_1 > 0)$. *Let $f(z)$ given by* (1) *belongs to* $\mathscr{K}^{j,k}(\alpha,\beta,\phi)$ *and σ_3 given by*

$$\sigma_3 = \frac{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))^2}{B_1(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))}\left[\frac{B_2}{B_1} + B_1 \frac{\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))}\right].$$

If $\sigma_1 < \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{1}{B_1^2}\left[(B_1 - B_2)\frac{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))^2}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))} - B_1^2 \frac{\psi_2\chi_2(\alpha,\beta)(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))} + \mu B_1^2\right]|a_2|^2$$

$$\le \frac{B_1}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))}.$$

If $\sigma_3 < \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{1}{B_1^2}\left[(B_1 + B_2)\frac{(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))^2}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))} + B_1^2 \frac{\psi_2\chi_2(\alpha,\beta)(\delta_2(\alpha,\beta) - \psi_2\chi_2(\alpha,\beta))}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))} - \mu B_1^2\right]|a_2|^2$$

$$\le \frac{B_1}{(\delta_3(\alpha,\beta) - \psi_3\chi_3(\alpha,\beta))},$$

where $\psi_n, \delta_n(\alpha,\beta)$ and $\chi_n(\alpha,\beta)$ are defined respectively by (2),(4) *and* (5).

Remark 2.6. *(i)For $\alpha=\beta=0, j=1$ we get the result obtained by by Al-Shaqsi and Darus* [1].
(ii)For $\alpha=\beta=0, j=1, k=2$ we obtain the result obtained by Shanmugam et al. [13].
(iii)For $\alpha=\beta=0, j=1, k=1$ we arrive to the result obtained by Ma and Minda [2].

By Lemma 2.1, we can obtain the following theorem.

Theorem 2.7. *Let $\phi(z)=1+B_1z+B_2z^2+\ldots(B_1>0)$. If $f(z)$ given by* (1) *belongs to $\mathscr{K}^{j,k}(\alpha,\beta,\phi)$, then*

$$|a_3-\mu a_2^2| \leq \frac{B_1}{2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\times \max\left\{1,\left|\frac{B_2}{B_1}+\frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}-\mu\frac{B_1(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right|\right\},$$

where $\psi_n, \delta_n(\alpha,\beta)$ and $\chi_n(\alpha,\beta)$ are defined respectively by (2),(4) *and* (5).
The result is sharp.

3. Application to Functions Defined by Fractional Derivatives

The fractional derivatives of order γ in the sense of Riemann Liouville is defined [5] by

$$D_z^\gamma f(z)=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^\gamma}d\zeta.\ 0\leq\gamma<1,$$

where f is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\gamma}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Fractional derivative of higher order are defined by

$$D_z^{\gamma+\varsigma}f(z)=\frac{d^\varsigma}{dz^\varsigma}D_z^\gamma f(z),\ \varsigma\in\mathbb{N}_0.$$

Using the fractional derivatives $D_z^\gamma f$ Owa and Srivastava [3] introduced the operator $\Omega^\gamma:\mathscr{A}\to\mathscr{A}$, which is known as an extension of fractional derivative and fractional integral as follows

$$\Omega^\gamma f(z)=\Gamma(2-\gamma)z^\gamma D_z^\gamma f(z),\ \gamma\neq 2,3,4,\ldots \tag{11}$$

The class $\mathscr{K}_{\gamma}^{j,k}(\alpha,\beta,\phi)$ consists of functions $f \in \mathscr{A}$ for which $\Omega^{\gamma} f \in \mathscr{K}^{j,k}(\alpha,\beta,\phi)$. The class $\mathscr{K}_{\gamma}^{j,k}(\alpha,\beta,\phi)$ is a special case of the class $\mathscr{K}_{g}^{j,k}(\alpha,\beta,\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} a_n z^k,\ z \in \mathscr{U}. \tag{12}$$

Now applying Theorem 2.3 for the function $(f*g)(z) = z + g_2a_2z^2 + g_3a_3z^3 + \ldots$ we get the following theorem.

Theorem 3.1. *Let $g(z) = 1 + g_1z + g_2z^2 + \ldots (g_n > 0)$. If $f(z)$ given by* (1) *belongs to $\mathscr{K}_{g}^{j,k}(\alpha,\beta,\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{g_3(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[B_2 + \frac{B_1^2\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))} - \mu\frac{g_3}{g_2^2}\frac{B_1^2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right] & \mu \leq \tau_1, \\ \frac{B_1}{g_3(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}, & \tau_1 \leq \mu \leq \tau_2, \\ -\frac{1}{g_3(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[B_2 + \frac{B_1^2\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))} - \mu\frac{g_3}{g_2^2}\frac{B_1^2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right], & \mu \geq \tau_2. \end{cases}$$

where

$$\tau_1 = \frac{g_2^2(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}{g_3B_1(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[-1 + \frac{B_2}{B_1} + \frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}\right],$$

and

$$\tau_2 = \frac{g_2^2(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}{g_3B_1(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[1 + \frac{B_2}{B_1} + \frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}\right],$$

where $\psi_n, \delta_n(\alpha,\beta)$ and $\chi_n(\alpha,\beta)$ are defined respectively by (2),(4) *and* (5).

The result is sharp.

Remark 3.2. *(i)Putting $\alpha = \beta = 0, j = 1$ we obtain the result by Al-Shaqsi and Darus* [1].
(ii)Putting $\alpha = \beta = 0, j = 1, k = 2$ we arrive to the result by Shanmugam et al [13] .

Since

$$\Omega^{\gamma} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} a_n z^k,\ z \in \mathscr{U},$$

we have

$$g_2 = \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma}, \tag{13}$$

and

$$g_3 = \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}. \tag{14}$$

For g_2, g_3 given by (13) and (14), respectively, Theorem 3.1 reduce the following theorem.

Theorem 3.3. *Let $\gamma < 2$. If $f(z)$ given by (1) belongs to $\mathscr{K}_\gamma^{j,k}(\alpha,\beta,\phi)$, then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{(2-\gamma)(3-\gamma)}{6(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[B_2+\frac{B_1^2\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}-\mu\frac{3(2-\gamma)}{2(3-\gamma)}\frac{B_1^2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right] & \mu \leq \tau_1^*, \\ \frac{(2-\gamma)(3-\gamma)B_1}{6(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}, & \tau_1^* \leq \mu \leq \tau_2^*, \\ -\frac{(2-\gamma)(3-\gamma)}{6(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[B_2+\frac{B_1^2\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}-\mu\frac{3(2-\gamma)}{2(3-\gamma)}\frac{B_1^2(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}\right], & \mu \geq \tau_2^*. \end{cases}$$

where

$$\tau_1* = \left(\frac{2(3-\gamma)}{3(2-\gamma)}\right)\frac{g_2^2(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}{g_3B_1(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[-1+\frac{B_2}{B_1}+\frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}\right],$$

and

$$\tau_2^* = \left(\frac{2(3-\gamma)}{3(2-\gamma)}\right)\frac{g_2^2(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))^2}{g_3B_1(\delta_3(\alpha,\beta)-\psi_3\chi_3(\alpha,\beta))}\left[1+\frac{B_2}{B_1}+\frac{B_1\psi_2\chi_2(\alpha,\beta)}{(\delta_2(\alpha,\beta)-\psi_2\chi_2(\alpha,\beta))}\right],$$

where $\psi_n, \delta_n(\alpha,\beta)$ and $\chi_n(\alpha,\beta)$ are defined respectively by (2),(4) *and* (5).
The result is sharp.

Remark 3.4. *(i) Putting $\alpha=\beta=0, j=1$ we get the result by Al-Shaqsi and Darus* [1].
(ii) Putting $\alpha=\beta=0, j=1, k=2$ we obtain the result by Shanmugam et al [13].

4. Conclusion

A modest attempt has been made in this paper to introduce the class $\mathscr{K}^{j,k}(\alpha,\beta,\phi)$ which provides an interesting transition from of starlike and convex functions by combining the concept of (j,k)-symmetrical functions. We derived the sharp upper bounds of $|a_3-\mu a_2^2|$ for functions belonging to the subclass $\mathscr{K}^{j,k}(\alpha,\beta,\phi)$. There is further scope to improve using the generalized the class and symmetric functions.

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