Innovative new Results on Prime Graphs

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ABSTRACT

We investigate prime labeling for some graphs resulted by identifying any two vertices of some graphs. We also introduce the concept of strongly prime graph and prove that the graphs C_n, P_n, and $K_{1,n}$ are strongly prime graphs. Moreover we prove that W_n is a strongly prime graph for every even integer $n \geq 4$.

Keywords: Prime Labeling; Prime Graph; Strongly Prime Graph

1. Introduction

We begin with finite, undirected and non-trivial graph $G = (V(G), E(G))$ with the vertex set $V(G)$ and the edge set $E(G)$. Throughout this work C_n denotes the cycle with n vertices and P_n denotes the path on n vertices. In the wheel $W_n = C_n + K_1$ the vertex corresponding to K_1 is called the apex vertex and the vertices corresponding to C_n are called the rim vertices, where $n \geq 3$. The star $K_{1,n}$ is a graph with one vertex of degree n called apex and n vertices of degree one are called pendant vertices. Throughout this paper $|V(G)|$ and $|E(G)|$ denote the cardinality of vertex set and edge set respectively.

For various graph theoretic notation and terminology we follow Gross and Yellen [1] while for number theory we follow Burton [2]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1: If the vertices are assigned values subject to certain condition(s) then it is known as *graph labeling*.

Vast amount of literature is available in printed as well as in electronic form on different kind of graph labeling problems. For a dynamic survey of graph labeling problems along with extensive bibliography we refer to Gallian [3].

Definition 1.2: A *prime labeling* of a graph G is an injective function $f: V(G) \rightarrow \{1,2,3,\cdots,|V(G)|\}$ such that for every pair of adjacent vertices u and v, $gcd\left(f(u), f(v)\right) = 1$. The graph which admits a prime labeling is called a *prime graph*.

The notion of a prime labeling was originated by Entringer and was discussed in a paper by Tout *et al.* [4]. Many researchers have studied prime graphs. For e.g. Fu and Huang [5] have proved that P_n and $K_{1,n}$ are prime graphs. Lee *et al.* [6] have proved that W_n is a prime graph if and only if n is even. Deretsky *et al.* [7] have proved that C_n is a prime graph.

Definition 1.3: Let u and v be two distinct vertices of a graph G. A new graph $G_{u,v}$ is constructed by *identifying* (*fusing*) two vertices u and v by a single new vertex x such that every edge which was incident with either u or v in G is now incident with x in $G_{u,v}$.

Vaidya and Kanani [8] have established that the graph obtained by identifying any two vertices u and v (with $d(u, v) \geq 3$) of C_n ($n \geq 4$) is a prime graph. The switching invariance of various prime graphs is discussed by Vaidya and Prajapati [9]. In the present paper we investigate further results on prime graphs.

Bertrand's Postulate 1.4: For every positive integer $n > 1$ there is a prime p such that $n < p < 2n$.

2. Prime Labeling of Some Graphs

Theorem 2.1: The graph obtained by identifying any two vertices of $K_{1,n}$ is a prime graph.

Proof: The result is obvious for $n = 1, 2$. Therefore we start with $n \geq 3$. Let v_0 be the apex vertex and $v_1, v_2, \cdots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Due to the nature of $K_{1,n}$ two vertices can be identified in following two possible ways:

Case 1: The apex vertex v_0 is identified with any of the pendant vertices (say v_1). Let the new vertex be u_0 and the resultant graph be G.

Then $\deg(v_i) = 1$, for $i = 2, 3, \cdots, n$ and $\deg(u_0) = n+1$ as there is a loop incident at u_0. Define $f: V(G) \rightarrow \{1,2,\cdots,n\}$ as $f(v_i) = i$ for $i = 2, 3, \cdots, n$ and $f(u_0) = 1$. Obviously f is an injection and $gcd\left(f(u), f(v)\right) = 1$ for every pair of adjacent vertices

u and v of G. Hence G is a prime graph.

Case 2: Any two of the pendant vertices (say v_{n-1} and v_n) are identified. Let the new vertex be u_{n-1} and the resultant graph be G. So in G, $\deg(v_i) = 1$, for $i = 1, 2, \cdots, n-2$, $\deg(u_{n-1}) = 2$ and $\deg(v_0) = n$. Define $f : V(G) \to \{1, 2, \cdots, n\}$ as $f(v_i) = i+1$ for $i = 0, 1, 2, \cdots, n-2$ and $f(u_{n-1}) = n$. Obviously f is an injection and $gcd(f(u), f(v)) = 1$ for every pair of adjacent vertices u and v of G. Hence G is a prime graph.

Illustration 2.2: The prime labeling of the graph obtained by identifying the apex vertex with a pendant vertex of $K_{1,7}$ is shown in **Figure 1**.

Illustration 2.3: The prime labeling of the graph obtained by identifying two of the pendant vertices of $K_{1,7}$ is shown in **Figure 2**.

Theorem 2.4: If p is a prime and G is a prime graph of order p then the graph obtained by identifying two vertices with label 1 and p is also a prime graph.

Proof: Let f be a prime labeling of G and i be the label of the vertex v_i for $i = 1, 2, \cdots, p$. Moreover u_1 be the new vertex of the graph G_1 which is obtained by identifying v_1 and v_p of G. Define

$f_1 : \{u_1, v_2, v_3, \cdots, v_{p-1}\} \to \{1, 2, \cdots, p-1\}$ as

$$f_1(x) = \begin{cases} f(v_i) & \text{if } x = v_i,\ i = 2, 3, \cdots, p-1 \\ 1 & \text{if } x = u_1. \end{cases}$$

Then
$$f_1(x) = \begin{cases} i & \text{if } x = v_i,\ i = 2, 3, \cdots, p-1 \\ 1 & \text{if } x = u_1. \end{cases}$$

Obviously f_1 is an injection. For an arbitrary edge $e = uv$ of G_1 we claim that $gcd(f_1(u), f_1(v)) = 1$. To prove our claim the following cases are to be considered.

Case 1: If $u = u_1$ then $gcd(f_1(u), f_1(v)) = gcd(f_1(u_1), f_1(v)) = gcd(1, f_1(v)) = 1$.

Case 2: If $u \neq u_1$ and $v = u_1$ then $gcd(f_1(u), f_1(v)) = gcd(f_1(u), f_1(u_1)) = gcd(f_1(u), 1) = 1$.

Case 3: If $u \neq u_1$ and $v \neq u_1$ then $u = v_i$, $v = v_j$ for some $i, j = 2, 3, \cdots, p-1$ with $i \neq j$ then $gcd(f_1(u), f_1(v)) = gcd(f_1(v_i), f_1(v_j)) = gcd(f(v_i), f(v_j)) = 1$ as v_i and v_j are adjacent vertices in the prime graph G with the prime labelling f. Thus in all the possibilities f_1 admits a prime labeling for G_1. Hence G_1 is a prime graph.

Illustration 2.5: In the following **Figures 3** and **4** prime labeling of a graph G of order 5 and the prime labeling for the graph G_1 obtained by identifying the vertices of G with label 1 and 5 are shown.

Theorem 2.6: The graph obtained by identifying any two vertices of P_n is a prime graph.

Proof: Let $v_1, v_2, \cdots, v_n$ be the vertices of P_n. Let u be the new vertex of the graph G obtained by identifying two distinct vertices v_a and v_b of P_n. Then G is nothing but a cycle (possibly loop) with at the most two

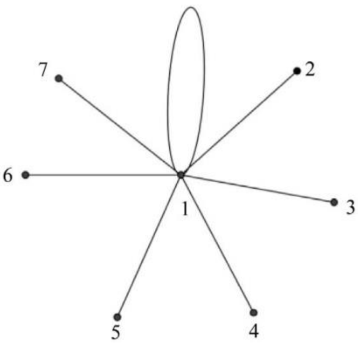

Figure 1. The prime labeling of the graph obtained by identifying the apex vertex with a pendant vertex in $K_{1,7}$.

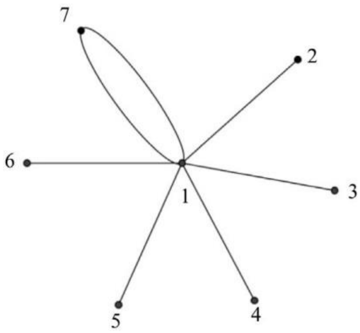

Figure 2. The prime labeling of the graph obtained by identifying two of the pendant vertices in $K_{1,7}$.

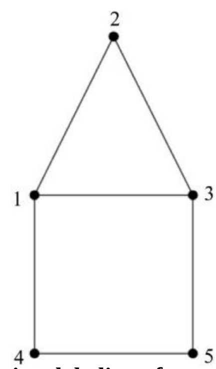

Figure 3. The prime labeling of a graph of order five.

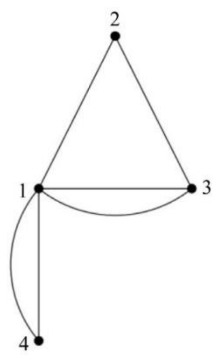

Figure 4. The prime labeling of the graph obtained by identifying the vertices of Figure 3 with label 1 and 5.

paths attached at u. Such graph is a prime graph as proved in Vaidya and Prajapati [10].

Illustration 2.7: In the following **Figures 5-9** prime labelings for P_5 and the graphs obtained by identifying two vertices in various possible ways are shown.

3. Strongly Prime Graphs

Definition 3.1: A graph G is said to be a *strongly prime graph* if for any vertex v of G there exits a prime labeling f satisfying $f(v) = 1$.

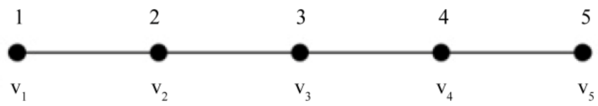

Figure 5. A prime labeling of P_5.

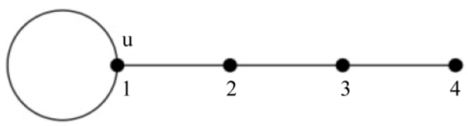

Figure 6. A prime labeling of the graph obtained by identifying v_1 and v_2 of P_5 of Figure 5.

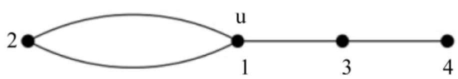

Figure 7. A prime labeling of the graph obtained by identifying v_1 and v_3 of P_5 of Figure 5.

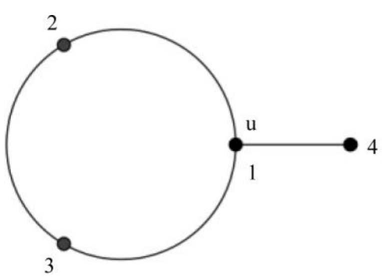

Figure 8. A prime labeling of the graph obtained by identifying v_1 and v_4 of P_5 of Figure 5.

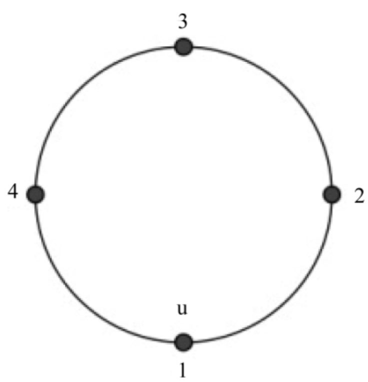

Figure 9. A prime labeling of the graph obtained by identifying v_1 and v_5 of P_5 of Figure 5.

Observation 3.2: K_3 is a strongly prime graph as any vertex of K_3 can be assigned label 1 and the remaining vertices can be assigned label 2 and 3 as shown in **Figure 10**.

Observation 3.3: If e is an edge of K_4 then $G = K_4 - e$ is a prime graph (see **Figure 11**) but it is not a strongly prime graph. If possible either of the vertices of G with degree 2 can be assigned the label 1. Suppose v_1 is assigned the label 1 then available vertex labels for the remaining three vertices of $G - v_1$ are 2, 3 and 4. Consequently the vertices corresponding to the labels 2 and 4 are adjacent in $G - v_1$. See **Figure 12**.

Observation 3.4: Every spanning subgraph of a strongly prime graph is a strongly prime graph. Because every spanning subgraph of a prime graph is a prime graph as proved by Seoud and Youssef [11].

Theorem 3.5: Every path is a strongly prime graph.

Proof: Let $v_1, v_2, \cdots, v_n$ be the consecutive vertices of P_n. If v_a is any arbitrary vertex of P_n then we have the following possibilities:

Case 1: If v_a is either of the pendant vertices (say $v_a = v_1$) then the function $f: V(P_n) \rightarrow \{1, 2, \cdots, n\}$ defined by $f(v_i) = i$, for all $i = 1, 2, \cdots, n$, is a prime labeling for P_n with $f(v_a) = f(v_1) = 1$.

Case 2: If v_a is not a pendant vertex then $a = j$ for some $j \in \{2, 3, \cdots, n-1\}$ then the function $f: V(P_n) \rightarrow \{1, 2, \cdots, n\}$ defined by

$$f(v_i) = \begin{cases} n + i - j + 1 & \text{if } i = 1, 2, \cdots, j-1; \\ i - j + 1 & \text{if } i = j, j+1, \cdots, n, \end{cases}$$

is a prime labeling with $f(v_a) = f(v_j) = 1$.

Thus from the cases described above P_n is a strongly prime graph.

Illustration 3.6: It is possible to assign label 1 to arbitrary vertex of P_5 in order to obtain different prime labeling as shown in **Figures 13-18**.

Theorem 3.7: Every cycle is a strongly prime graph.

Proof: Let $v_1, v_2, \cdots, v_n$ be the consecutive vertices of C_n. Let v be an arbitrary vertex of C_n. Then $v = v_j$ for some $j \in \{1, 2, \cdots, n\}$. The function $f: V(C_n) \rightarrow \{1, 2, \cdots, n\}$ defined by

$$f(v_i) = \begin{cases} n + i + 1 - j & \text{if } i = 1, 2, \cdots, j-1; \\ i - j + 1 & \text{if } i = j, j+1, \cdots, n, \end{cases}$$

is a prime labeling for C_n with $f(v) = f(v_j) = 1$. Thus f admits prime labeling as well as it is possible to assign label 1 to any arbitrary vertex of C_n. That is C_n, is strongly prime graph.

Theorem 3.8: $K_{1,n}$ is a strongly prime graph.

Proof: For $n = 1, 2$ the respective graphs P_2 and P_3 are strongly prime graphs as proved in the Theorem 3.5.

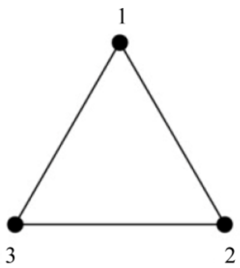

Figure 10. A strongly prime labeling of K_3.

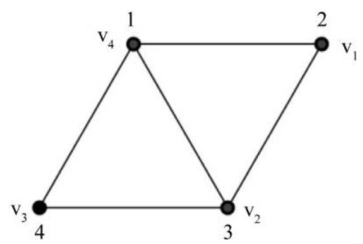

Figure 11. A prime labeling of $K_4 - e$.

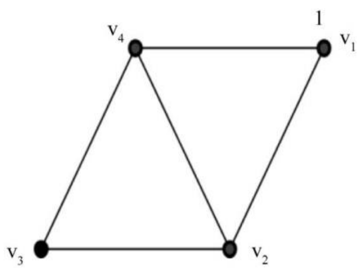

Figure 12. $K_4 - e$ is not a strongly prime graph.

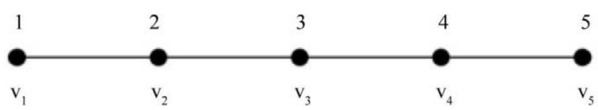

Figure 13. A prime labeling of P_5 having v_1 as label 1.

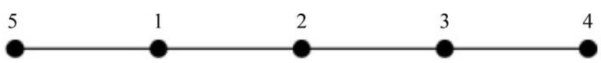

Figure 14. A prime labeling of P_5 having v_2 as label 1.

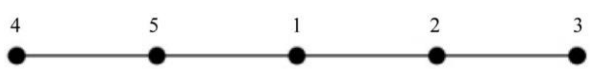

Figure 15. A prime labeling of P_5 having v_3 as label 1.

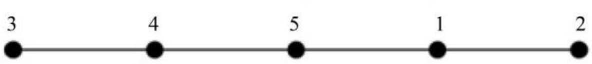

Figure 16. A prime labeling of P_5 having v_4 as label 1.

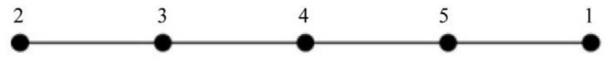

Figure 17. A prime labeling of P_5 having v_5 as label 1.

Figure 18. A prime labeling of P_5 having v_5 as label 1.

For $n > 2$ let v_0 be the apex vertex and $v_1, v_2, \cdots, v_n$ be the pendant vertices of $K_{1,n}$.

If x is any arbitrary vertex of $K_{1,n}$ then we have the following possibilities:

Case 1: If x is the apex vertex then $x = v_0$. Then the function $f : V(K_{1,n}) \to \{1, 2, \cdots, n+1\}$ defined by $f(v_i) = i+1$ for $i = 0, 1, 2, \cdots, n$ is a prime labeling on $K_{1,n}$ with $f(x) = f(v_0) = 1$.

Case 2: If x is one of the pendant vertices then $x = v_j$ for some $j \in \{1, 2, \cdots, n\}$. Define $f : V \to \{1, 2, \cdots, n, n+1\}$, $f(v_0) = p$, where p is the largest prime less than or equal to n and the remaining $n-1$ vertices are distinctly labeled from $\{2, \cdots, n, n+1\} - \{p\}$. According to Bertrand's postulate $\left\lfloor \frac{n+1}{2} \right\rfloor < p < n+1$. Therefore p is co-prime to every integer from $\{1, 2, \cdots, n, n+1\} - \{p\}$. Thus every edge $e = ab$ is incident to the apex vertex v_0 whose label is p, thus $gcd(f(a), f(b)) = gcd(p, f(b)) = 1$ or $gcd(f(a), f(b)) = gcd(f(a), p) = 1$. Hence this function f admits a prime labeling on $K_{1,n}$ with $f(x) = f(v_j) = 1$.

Thus from all the cases described above $K_{1,n}$ is a strongly prime graph.

Illustration 3.9: It is possible to assign label 1 to arbitrary vertex of $K_{1,7}$ in order to obtain prime labeling as shown in **Figures 19** and **20**.

Theorem 3.10: W_n is a strongly prime graph for every even positive integer $n \geq 4$.

Proof: Let v_0 be the apex vertex and $v_1, v_2, \cdots, v_n$ be the consecutive rim vertices of W_n. Let x be an arbitrary vertex of W_n. We have the following possibilities:

Case 1: x is the apex vertex of W_n that is $x = v_0$. Then the function $f : V(W_n) \to \{1, 2, \cdots, n+1\}$ defined as

$$f(v_i) = i+1, i = 0, 1, 2, \cdots, n. \quad (1)$$

Obviously f is an injection. For an arbitrary edge $e = ab$ of G we claim that $gcd(f(a), f(b)) = 1$. To prove our claim the following subcases are to be considered.

Subcase (1): if $e = v_i v_{i+1}$ for some $i \in \{1, 2, \cdots, n-1\}$

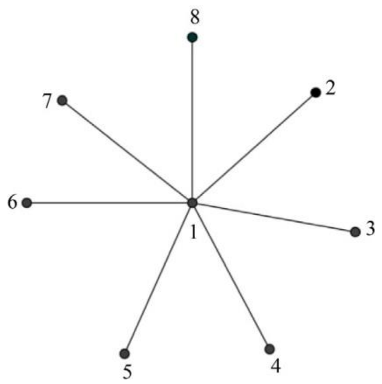

Figure 19. A prime labeling of $K_{1,7}$ with the apex vertex as label 1.

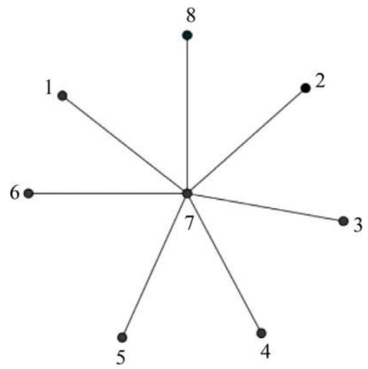

Figure 20. A prime labeling of $K_{1,7}$ with a pendant vertex as label 1.

then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd\,(i+1,\, i+1+1) =$ $gcd(i+1, i+2) = 1$ as $i+1$ and $i+2$ are consecutive positive integers.

Subcase (2): if $e = v_n v_1$ then $gcd\left(f(v_n), f(v_1)\right) =$ $gcd(n+1, 2) = 1$ as $n+1$ is an odd integer and it is not divisible by 2.

Subcase (3): if $e = v_0 v_i$ for some $i \in \{1, 2, \cdots, n\}$ then $gcd\left(f(v_0), f(v_i)\right) = gcd(1, i+1) = 1$.

Case 2: x is one of the rim vertices. We may assume that $x = v_{p-1}$ where p is the largest prime less than or equal to $n+1$. According to the Bertrand's Postulate such a prime p exists with $\left\lceil \frac{n+1}{2} \right\rceil < p < n+1$. Define a function $f : V(W_n) \rightarrow \{1, 2, 3, \cdots, n+1\}$ as

$$f(v_i) = \begin{cases} p, & \text{if } i = 0; \\ i+1, & \text{if } i \in \{1, 2, \cdots, n\} - \{p-1\}; \\ 1, & \text{if } i = p-1. \end{cases} \qquad (2)$$

The only difference between the definition of labeling functions of (1) and (2) is the labels 1 and p are interchanged. Then clearly f is an injection.

For an arbitrary edge $e = ab$ of G we claim that $gcd\left(f(a), f(b)\right) = 1$. To prove our claim the following subcases are to be considered.

Subcase (2): If $e = v_0 v_i$ for some $i \in \{1, 2, \cdots, n\}$ then $gcd\left(f(v_0), f(v_i)\right) = gcd\left(p, f(v_i)\right) = 1$ as p is co-prime to every integer from $\{1, 2, \cdots, n+1\} - \{p\}$.

Subcase (2): If $e = v_i v_{i+1}$ for some $i \in \{1, 2, \cdots, p-3\} \cup \{p, p+1, \cdots, n\}$ then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd(i+1, i+2) = 1$ as $i+1$ and $i+2$ are consecutive positive integers.

Subcase (3): If $e = v_i v_{i+1}$ for $i = p-2$ then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd\left(f(v_{p-2}), f(v_{p-1})\right) =$ $gcd\,(p-1, 1) = 1$.

Subcase (4): If $e = v_i v_{i+1}$ for $i = p-1$ then $gcd\left(f(v_i), f(v_{i+1})\right) = gcd\left(f(v_{p-1}), f(v_p)\right) =$ $gcd\,(1, p+1) = 1$

Thus in all the possibilities described above f admits prime labeling as well as it is possible to assign label 1 to any arbitrary vertex of W_n. That is, W_n is a strongly prime graph for every even positive integer $n \geq 4$.

Illustration 3.11: It is possible to assign label 1 to arbitrary vertex of W_8 in order to obtain prime labeling as shown in **Figures 21** and **22**.

Corollary 3.12: The friendship graph $C_3^{(n)}$ is a strongly prime graph.

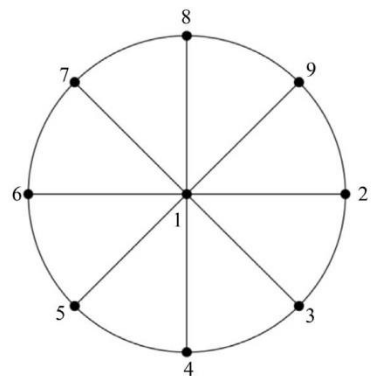

Figure 21. A prime labeling of W_8 with the apex vertex as label 1.

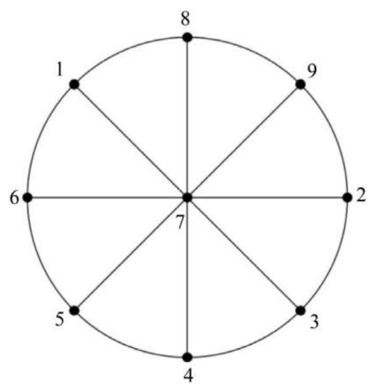

Figure 22. A prime labeling of W_8 with a rim vertex as label 1.

Proof: The friendship graph $C_3^{(n)}$ is a one point union of n copies of C_3. It can also be thought as a graph obtained by deleting every alternate rim edge of W_{2n}. Being a spanning subgraph of strongly prime graph W_{2n}, $C_3^{(n)}$ is a strongly prime graph.

Corollary 3.13: The star $K_{1,2n}$ is a strongly prime graph.

Proof: $K_{1,2n}$ is obtained from strongly prime graph W_{2n} by deleting all the rim edges of the W_{2n}. Being a spanning subgraph of strongly prime graph W_{2n}, $K_{1,2n}$ is a strongly prime graph.

4. Concluding Remarks

The prime numbers and their behaviour are of great importance as prime numbers are scattered and there are arbitrarily large gaps in the sequence of prime numbers. If these characteristics are studied in the frame work of graph theory then it is more challenging and exciting as well.

Here we investigate several results on prime graphs. This discussion becomes more interesting in the situation when two vertices of a graph are identified. We also introduce a concept of strongly prime graph. As every prime graph is not a strongly prime graph it is very exciting to investigate graph families which are strongly prime graphs. We investigate several classes of prime graph which are strongly prime graph.

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