ENRESDOWEDNESS OF CORONA PRODUCT GRAPHS

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Abstract

Let G = (V, E) be a non empty, finite, simple graph. A dominating set of a graph G containing a minimum dominating set of G is called a γ - endowed dominating set of G. If that set is of cardinality *k* then it is called a *k* γ – endowed dominating set. *k* - γ_r enresdowed graph is one in which every restrained dominating set of cardinality *k* contains a minimum restrained dominating set. In this paper we found the enresdowedness property for the corona product of a complete graph with a star.

Keywords : Enresdowed graphs, Star graph , Complete Graph

1. INTRODUCTION

Domination Theory has a wide range of applications to many fields like Engineering Communication Networks, Social sciences, linguistics and many others. Let G = (V, E) be a non empty, finite, simple graph. A subset D of V(G) is called a dominating set of G if for every $v \in V - D$, there exists $u \in D$ such that u and v are adjacent. The minimum cardinality of the dominating set is called the domination number and it is denoted by $\gamma(G)$[4]. The restrained dominating set of a graph is a dominating set in which every vertex in V – D is adjacent to some other vertex in V – D. The minimum cardinality of the restrained dominating set is called the restrained domination number and it is denoted by $\gamma_r(G)$[2]. A graph is said to be *k* - γ_r enresdowed graph if every restrained dominating set of cardinality *k* contains a minimum restrained dominating set[6].

Frucht and Harary [3] introduced a new product on two graphs G_1 and G_2 , called corona product denoted by $G_1 \odot G_2$. The object is to construct a new and simple operation on two graphs G_1 and G_2 called their corona , with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G_1 and G_2.

2. CORONA PRODUCT GRAPHS $K_n \odot K_{1,m}$ AND ITS PROPERTIES

Definition 2.1

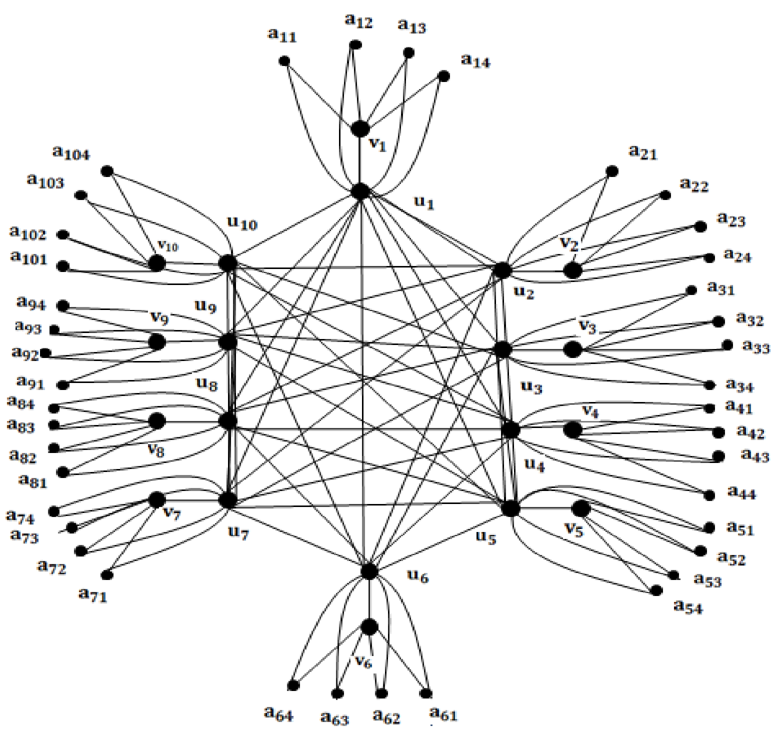
Let *k* be a positive integer. A simple, finite, non trivial graph G = (V, E) is called a *k* - γ_r enresdowed graph if every restrained dominating set of G of cardinality *k* contains a minimum restrained dominating set γ_r of G.[6]

Definition 2.2

The corona product of a complete graph K_n with a star graph $K_{1,m}$ for n ≥ 3and m ≥ 2, is a graph obtained by taking one copy of a n vertex complete graph K_n and n copies of $K_{1,m}$ and then joining the i th vertex of K_n to all vertices of i th copy of $K_{1,m}$ .This graph is denoted by $K_n \odot K_{1,m}$.

Example 2.3

Let $G = K_{10} \odot K_{1,4}$ be a graph with V(G) = $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{24}, a_{31}, a_{32}, a_{33}, a_{34}, a_{41}, a_{42}, a_{43}, a_{44}, a_{51}, a_{52}, a_{53}, a_{54}, a_{61}, a_{62}, a_{63}, a_{64}, a_{71}, a_{72}, a_{73}, a_{74}, a_{81}, a_{82}, a_{83}, a_{84}, a_{91}, a_{92}, a_{93}, a_{94}, a_{101}, a_{102}, a_{103}, a_{104}\}$



Theorem 2.3

Let G = $K_n \odot K_{1,m}$, for n ≥ 3 and m = 2 be a graph, on p vertices where p = n (m+2) , then G is k - γ_r enresdowed for any k, where $k = \gamma_r$ and n, except for some $k = \gamma_r + j$, $1 \leq j \leq mn + (n-1)$.

Proof

Given G is a graph, where G = $K_n \odot K_{1,m}$, for n ≥ 3 and m = 2. Let G be a graph on p vertices, where p = n (m+2) and D be the γ_r set of G.

Let $C_1 = \{u_1, u_2, \ldots, u_l, \ldots, u_r\}$, $1 \leq l \leq r$ be the vertices of the complete graph K_n in G, $C_2 = \{v_1, v_2, \ldots, v_l, \ldots, v_r\}$, $1 \leq l \leq r$ be the set of first partition vertices of G, where each vertex in the set $\{v_l\}$, $1 \leq l \leq r$ is adjacent to its corresponding vertex in the set $\{u_l\}$, $1 \leq l \leq r$ and $C_3 = \{b_1, w_1, b_2, w_2, \ldots, b_l, w_l, \ldots, b_r, w_r\}$, $1 \leq l \leq 2r$ be the set of second partition vertices of G, where each set of vertices $\{b_l w_l\}$, $1 \leq l \leq 2r$ is adjacent to $\{v_l\}$, $1 \leq l \leq r$ of first partition vertices of G, and $\{u_l\}$, $1 \leq l \leq r$ of vertices in K_n graph of G.

Choose the vertex v_1 for the γ_r set of G, then the vertices b_1, w_1, u_1 are dominated , then choose the vertex v_2 for the γ_r set D of G, then the vertices b_2, w_2, u_2 are dominated. Without loss of generality, choose the vertex v_l, $1 \leq l \leq r$, then the vertices b_l, w_l, u_l are dominated. Finally choose the vertex v_r, for the γ_r set D of G then the vertices b_r, w_r, u_r are dominated. Thus the set $D = \{v_1, v_2, \ldots, v_l, \ldots, v_r\}$, $1 \leq l \leq r$ forms the minimum restrained dominating set of cardinality k, where $k = \gamma_r$. Hence G is k - γ_r enresdowed for any k, where $k = \gamma_r$.

Consider the restrained dominating set D_1 of cardinality k_1 , where $k_1 = \gamma_r + 1$. Let $D_1 = (D - \{v_r\}) \cup \{b_r, w_r\}$, where $D_1 = \{v_1, v_2, \ldots, v_l, \ldots, v_{r-1}, b_r, w_r\}$ and $V - D_1 = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_r, b_1, w_1, b_2, w_2, \ldots, b_l, w_l, \ldots, b_{r-1}, w_{r-1}\}$, since each $\{u_l\}$ is adjacent with $\{b_l, w_l\}$, for $1 \leq l \leq r$ and also with v_r, hence there exist no isolate vertex in $V - D_1$. Hence D_1 forms a restrained dominating set of cardinality k_1, where $k_1 = \gamma_r + 1$, which does not contain the minimum restrained dominating set D. Thus G is not k_1- γ_r enresdowed.

Consider the restrained dominating set D_2 of cardinality k_2 , where $k_2 = \gamma_r + 2$. Let $D_2 = D_1 \cup \{w_{r-1}\}$,where $D_2 = \{v_1, v_2, \ldots, v_l, \ldots, v_{r-1}, w_{r-1}, b_r, w_r\}$ and $V - D_2 = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_r, b_1, w_1, b_2, w_2, \ldots, b_l, w_l, \ldots, b_{r-1}\}$ then D_2 forms a restrained dominating set of cardinality k_2 , where $k_2 = \gamma_r + 2$, in which there exist no γ_r set D. Thus G is not k_2 - γ_r enresdowed. Similarly consider the restrained dominating set D_3 of cardinality k_3, where $k_3 = \gamma_r + 3$. Let $D_3 = D_2 \cup \{b_{r-1}\}$, where $D_3 = \{v_1, v_2, \ldots, v_l, \ldots, v_{r-1}, b_{r-1}, w_{r-1}, b_r, w_r\}$and $V - D_3 = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_r, b_1, w_1, b_2, w_2, \ldots, b_l, w_l, \ldots, b_{r-2}, w_{r-2}\}$. Thus D_3 forms a restrained dominating set of cardinality k_3, where $k_3 = \gamma_r + 3$ and there exist no γ_r set D in D_3. Hence G is not k_3 - γ_r enresdowed.

Proceeding similarly, consider a set, D_{r-1} where $D_{r-1} = \{ u_1, u_2, \ldots, u_l, \ldots, u_r, v_1, v_2, \ldots, v_l, \ldots, v_r, b_1, w_1, b_2, w_2, \ldots, b_{l-1}, w_{l-1}, w_l, \ldots, b_r, w_r \}$ and $V - D_{r-1} = \{b_l\}$, $1 \leq l \leq r$ is of cardinality k_{r-1}, where k_{r-1} = n (m+2) – 1. Thus D_{r-1} is not a restrained dominating set and G is not k_{r-1}- γ_r enresdowed. Hence G is not k - γ_r enresdowed for any k , $k = \gamma_r + j$, $1 \leq j \leq mn + (n - 1)$. Consider the set $D_r = D_{r-1} \cup \{b_l\}$, $1 \leq l \leq r$, which is of cardinality k_r, where $k_r = p = n\,(m+2)$. Thus the set D_r contains the minimum restrained dominating set D. Hence G is k_r - γ_r enresdowed.

Theorem 2.4

The graph $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ is a connected graph.

Proof

Consider the graph $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$. Since each vertices of K_n, $n \geq 3$ is adjacent to every other vertices of $K_{1,m}$. Thus the vertices of K_n are connected to the vertices of $K_{1,m}$. Since K_n and $K_{1,m}$ are connected, it follows that the graph G is connected . Hence $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ is a connected graph.

Theorem 2.5

Let $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ be a connected graph, then the degree of a vertex v_i in G is given by

$$d(v_i) = \begin{cases} n + m & \text{if } v_i \in K_n \text{, for } 1 \leq i \leq n \\ m + 1 & \text{if } v_i \in K_{1,m} \text{ and } v_i \text{, for } 1 \leq i \leq n, \text{ in the first partition} \\ 2 & \text{if } v_i \in K_{1,m} \text{ and } v_i \text{, for } 1 \leq i \leq mn, \text{ in the second partition} \end{cases}$$

Proof

Consider the graph $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$. Since each vertex in $\{v_i\}$, $1 \leq i \leq n$ of K_n is adjacent to m +1 vertices of its corresponding i^{th} copy of $K_{1,m}$, $m \geq 2$ in G. Then the degree of the vertex v_i, $1 \leq i \leq n$ of K_n is $d(v_i) = n - 1 + m + 1 = n + m$.

Consider any vertex from $\{v_i\}$, $1 \leq i \leq n$, from the first partition vertices of $K_{1,m}$, then each vertex v_i, $1 \leq i \leq n$ are adjacent only to its corresponding i^{th} copy vertex in K_n and its corresponding m vertices in the second partition vertices of $K_{1,m}$. Thus $d(v_i) = m + 1$ for any vertex v_i which belongs to the first partition of $K_{1,m}$. Since every vertex $\{ v_i \}$, $1 \leq i \leq mn$ which belongs to the second partition vertices of $K_{1,m}$ is adjacent only to the i^{th} copy of $\{v_i\}$, $1 \leq i \leq n$ in the first partition of $K_{1,m}$ and adjacent to its i^{th} copy vertex $\{v_i\}$, $1 \leq i \leq n$ in K_n . Hence $d(v_i) = 2$, $1 \leq i \leq mn$, for any v_i in the second partition vertices of $K_{1,m}$.

heorem 2.6

Let $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ be a connected graph then G is not well restrained dominated graph.

Proof

Given $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ is a connected graph on p vertices, where $p = n\,(m+2)$. Let $C_1 = \{u_1, u_2, \ldots., u_l, \ldots., u_r\}$, $1 \leq l \leq r$ be the vertex set of K_n in G, $C_2 = \{v_1, v_2, \ldots., v_l, \ldots., v_r\}$, $1 \leq l \leq r$ be the first partition vertex set of G, and $C_3 = \{a_{11}, a_{12}, \ldots., a_{1m}, a_{21}, a_{22}, \ldots., a_{2m}, \ldots., a_{l1}, a_{l2}, \ldots., a_{lm}, \ldots\ldots, a_{r1}, a_{r2}, \ldots., a_{rm}\}$, $1 \leq l \leq mr$ be the second partition vertex set of G. The set of vertices $\{a_{11}, a_{12}, \ldots., a_{1m}\}$ are attached to the vertices v_1 and u_1 in G and $\{a_{21}, a_{22}, \ldots., a_{2m}\}$ is attached to the vertices v_2 and u_2 in G. In general the set of vertices $\{a_{l1}, a_{l2}, \ldots., a_{lm}\}$ are attached to v_l and u_l in G and also finally the set of vertices $\{a_{r1}, a_{r2}, \ldots., a_{rm}\}$ are attached to v_r and u_r in G, such that $d(a_{lj}) = 2$, $1 \leq j \leq m$.

Consider the set of vertices $\{v_1, v_2, \ldots., v_l, \ldots., v_r\}$, $1 \leq l \leq r$ for the γ_r set D in G, thus $D = \{v_1, v_2, \ldots., v_l, \ldots., v_r\}$, $1 \leq l \leq r$ forms the minimal restrained dominating set of G, which is also minimum with cardinality k, where $k = \gamma_r$. Consider a restrained dominating set D_1 of cardinality k_1, where $k_1 > k$, $k_1 = \gamma_r - 1 + m$, for $m \geq 2$. Let $D_1 = \{v_1, v_2, \ldots., v_l, \ldots, v_{r-1}, a_{r1}, a_{r2}, \ldots., a_{rm}\}$ and $V - D_1 = \{u_1, u_2, \ldots., u_l, \ldots., u_r, v_r, a_{11}, a_{12}, \ldots., a_{1m}, a_{21}, a_{22}, \ldots., a_{2m}, \ldots., a_{l1}, a_{l2}, \ldots., a_{lm}, \ldots\ldots, a_{(r-1)1}, a_{(r-1)2}, \ldots., a_{(r-1)m}\}$, then there exist the following cases

Case(i)

Consider the set $D_s = D_1 - \{v_l\}$, $1 \leq l \leq r - 1$ then $D_s = \{v_1, v_2, \ldots., v_{l-1}, v_{l+1} \cdots, v_{r-1}, a_{r1}, a_{r2}, \ldots., a_{rm}\}$ and $V - D_s = \{u_1, u_2, \ldots., u_l, \ldots, u_r, v_l, v_r, a_{11}, a_{12}, \ldots., a_{1m}, a_{21}, a_{22}, \ldots., a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots., a_{lm}, \ldots\ldots, a_{(r-1)1}, a_{(r-1)2}, \ldots., a_{(r-1)m}\}$, Thus the set of vertices $\{a_{l1}, a_{l2}, \ldots., a_{lm}\}$ are not dominated by any vertices in D_s. Hence D_s is not a restrained dominating set.

Case(ii)

Consider the set $D_t = D_1 - \{a_{ri}\}$, $1 \leq i$,$r \leq m$ then $D_t = \{v_1, v_2, \ldots., v_l, \ldots, v_{r-1}, a_{r1}, a_{r2}, \ldots., a_{r(i-1)}, a_{r(i+1)}, \ldots\ldots, a_{rm}\}$, $1 \leq i \leq m$, and $V - D_t = \{u_1, u_2, \ldots., u_l, \ldots, u_r, v_r, a_{11}, a_{12}, \ldots., a_{1m}, a_{21}, a_{22}, \ldots., a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots., a_{lm}, \ldots\ldots, a_{(r-1)1}, a_{(r-1)2}, \ldots., a_{(r-1)m}, a_{ri}\}$. Since a_{ri} can be dominated only by either u_r or v_r, where both u_r and v_r belong to the $V - D_t$ set. Thus no vertex in D_t dominates a_{ri} in $V - D_t$ set. Thus D_t is not a restrained dominating set of G. Since there exist a minimal restrained dominating set D_1 of G with cardinality k_1, where $k_1 > k$ and $k_1 = \gamma_r - 1 + m$, for $m \geq 2$. Hence G is not well restrained dominated graph .

Theorem 2.7

Let $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ be a graph with the vertex set V(G) and edge set E(G). The cardinality of the vertex set and edge set of G is given by $|V(G)| = n(m+2)$ and $|E(G)| = \frac{1}{2}(n^2 + 4nm + n)$.

Proof

Given G is a graph, where $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$, whose vertex set is denoted by V(G) and edge set is denoted by E(G). By the definition of the graph $G = K_n \odot K_{1,m}$, G contains the n vertices of the complete graph K_n and the m + 1 vertices of $K_{1,m}$ in n – copies. Thus the cardinality of the set of all vertices of G is given by, $|V(G)| = n + n(m+1) = n(1 + m + 1) = n(m+2)$. By theorem 2.5, the degree of a vertex $v_i \in G$ is given by

$$d(v_i) = \begin{cases} n + m & \text{if } v_i \in K_n, \text{for } 1 \leq i \leq n \\ m + 1 & \text{if } v_i \in K_{1,m} \text{ and } v_i, \text{for } 1 \leq i \leq n, \text{ in the first partition} \\ 2 & \text{if } v_i \in K_{1,m} \text{ and } v_i, \text{for } 1 \leq i \leq mn, \text{in the second partition} \end{cases}$$

$$\text{Hence } |E(G)| = \frac{1}{2}\left(\sum_{v_i \in K_n} \deg(v_i) + n\sum_{v_i \in K_{1,m}} \deg(v_i)\right)$$

$$= \frac{1}{2}(n(n+m) + n(m+1) + 2nm)$$

$$= \frac{1}{2}(n^2 + nm + nm + n + 2nm)$$

$$|E(G)| = \frac{1}{2}(n^2 + 4nm + n).$$

Definition 2.8

A closed trail containing all vertices and edges is called an eulerian trail. A graph having an eulerian trail is called an eulerian graph.[1]

Theorem 2.9

The connected graph $G = K_n \odot K_{1,m}$, for odd n , $m \geq 3$ is eulerian.

Proof

Given $G = K_n \odot K_{1,m}$, be a connected graph, for odd n, $m \geq 3$. Let $C_1 = \{u_1, u_2, \ldots, u_l, \ldots, u_r\}$, $1 \leq l \leq r$ be the set of vertices of the complete graph K_n of G, $C_2 = \{v_1, v_2, \ldots, v_l, \ldots, v_r\}$, $1 \leq l \leq r$ be the set of first partition vertices of G and $C_3 = \{a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}, \ldots\ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$, $1 \leq l \leq mr$ be the set of second partition vertices of G, Thus the vertex set of G is $V(G) = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_1, v_2, \ldots, v_l, \ldots, v_r, a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}, \ldots\ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$ and the edge set of G is given by $E(G) = \{u_1u_2, u_1u_3, \ldots, u_1u_l, \ldots, u_1u_r, u_2u_3, u_2u_4, \ldots, u_2u_l, \ldots, u_2u_r, \ldots, u_lu_1, u_lu_2, \ldots,$

$u_{l-1}u_l$, u_lu_r , ….. , u_ru_1 , u_ru_2 , ….,u_ru_{r-1} , u_1v_1 , u_2v_2 , u_3v_3 , …. , u_lv_l , …. , u_rv_r , $a_{11}v_1$, $a_{12}v_1$,…., $a_{1m}v_1$, $a_{21}v_2$, $a_{22}v_2$,…., $a_{2m}v_2$, …., $a_{l1}v_l$, $a_{l2}v_l$,…., $a_{lm}v_l$,…., $a_{r1}v_r$, $a_{r2}v_r$,…., $a_{rm}v_r$, $a_{11}u_1$, $a_{12}u_1$,…., $a_{1m}u_1$, $a_{21}u_2$, $a_{22}u_2$,…., $a_{2m}u_2$,….., $a_{l1}u_l$, $a_{l2}u_l$,…., $a_{lm}u_l$, …., $a_{r1}u_r$, $a_{r2}u_r$,…., $a_{rm}u_r$}.

Consider the set C_3 of second partition vertices of G, without loss generality, choose any vertex a_{lj}, $1 \le l \le r$, $1 \le j \le m$ from the vertex set C_3, which is adjacent only with the vertex v_l and u_l , $1 \le l \le r$. Thus $d(a_{lj}) = 2$, $1 \le j \le m$. Hence the set of all vertices in the second partition vertex set is of even degree. Consider the set C_2 of first partition vertices of G. Without loss of generality , choose any vertex v_l, $1 \le l \le r$ from the vertex set C_2 , then the vertex v_l , $1 \le l \le r$ is adjacent to the vertex $u_l \in K_n$ and to the odd m vertices $\{a_{l1}, a_{l2},…., a_{lm}\}$. Thus v_l, is adjacent to the set of vertices $\{u_l, a_{l1}, a_{l2},…., a_{lm}\}$, which is of cardinality m+ 1, since m is odd , m + 1 is even. Thus every vertex v_l, $1 \le l \le r$ which belongs to the first partition vertex set C_2 is of even degree.

Finally consider the set of vertices $\{u_1, u_2,…., u_l, … , u_r\}$, $1 \le l \le r$ in the complete graph K_n, where n is odd . In K_n, $d(u_l) = r – 1$ for every u_l, $1 \le l \le r$. Thus the edges of the vertices in K_n is given by $d(u_l) = r – 1 + 1 + m = r + m = s$. Since r and m is odd, r + m is even, thus $d(u_l)$, $1 \le l \le r$ is even . Hence the set of all vertices in the complete graph K_n of G is of even degree. Thus the set {u_1, a_{11}, v_1, a_{1m}, u_1, v_1, $a_{1(m-1)}$, u_1, $a_{1(m-2)}$,v_1, $a_{1(m-3)}$, … , a_{12}, u_1, u_2, a_{21}, v_2 , a_{2m}, u_2, v_2, $a_{2(m-1)}$, u_2, $a_{2(m-2)}$, v_2, $a_{2(m-3)}$, …., a_{22},u_2, u_3, a_{31}, v_3 , a_{3m}, u_3, v_3, $a_{3(m-1)}$, u_3, $a_{3(m-2)}$, v_3, $a_{3(m-3)}$, …. , a_{32}, u_3, ….., u_l, a_{l1}, v_l, a_{lm}, u_l, v_l, $a_{l(m-1)}$, u_l, $a_{l(m-2)}$, v_l, $a_{l(m-3)}$, ….,a_{l2}, u_l, ….,u_r , a_{r1}, v_r , a_{rm}, u_r, v_r , $a_{r(m-1)}$, u_r , $a_{r(m-2)}$, v_r, $a_{r(m-3)}$, …. , a_{r2}, u_r, u_1} forms the eulerian cycle for the graph $G = K_n \odot K_{1,m}$, for odd n , m ≥3. Since every vertices in the vertex set of G is of even degree and it contains an eulerian cycle. Hence G is eulerian.

Theorem 2.10

Let $G = K_n \odot K_{1,m}$, $n \ge 3$, $m \ge 2$ be a connected graph and D be the γ_r set of G, then G has n - γ_r private neighbours with respect to the γ_r set D.

Proof

Given $G = K_n \odot K_{1,m}$, $n \ge 3$, $m \ge 2$ is a connected graph. Let $C_1 = \{u_1, u_2,…., u_l, … , u_r\}$, $1 \le l \le r$ be the vertex set of K_n, $C_2 = \{v_1, v_2,…., v_l, … , v_r\}$, $1 \le l \le r$ be the set of first partition vertices of G and $C_3 = \{a_{11}, a_{12},…., a_{1m}, a_{21}, a_{22},…., a_{2m},…., a_{l1}, a_{l2},…., a_{lm},……, a_{r1}, a_{r2},…., a_{rm}\}$, $1 \le l \le mr$ be the set of second partition vertices of G.

Let $D = \{v_1, v_2,…., v_l,…., v_r\}$, $1 \le l \le r$ forms the γ_r set of G. Thus the set D dominates every other vertices in the set C_1 and C_3 of G. Consider any vertex u_l, $1 \le l \le r$ from the complete graph K_n, $n \ge 3$, where u_l, $1 \le l \le r$ is adjacent only to the vertex $v_l \in D$, and to the vertices u_{l-1}, $u_{l+1} \in V – D$. Thus u_l, $1 \le l \le r$ forms a private neighbour of v_l with respect to

D and the set of vertices $\{u_1, u_2, \ldots, u_l, \ldots, u_r\}$, $1 \le l \le r$ forms the private neighbourhood set of G with respect to D.

Consider any vertex a_{lj}, $1 \le l \le r$, $1 \le j \le m$ from the second partition vertex set of G. Then each a_{lj}, $1 \le l \le r$, $1 \le j \le m$ is adjacent only to the vertex $v_l \in D$ and $u_l \in V - D$. Thus a_{lj} forms a private neighbour of v_l with respect to D. Hence the set of vertices $\{a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$, $1 \le l \le mr$ also forms the private neighbour set of G with respect to D. Thus the cardinality of the set of all private neighbours of G with respect to D = cardinality of $\{u_l\}$ + cardinality of $\{a_{lj}\}$ = r + mr = r (m + 1) = n - γ_r. Therefore G has n - γ_r private neighbours with respect to D.

Theorem 2.11

Let $G = K_n \odot K_{1,m}$, $n \ge 3$, $m > 2$ be a connected graph . Let D be the γ_r set of G, then G is not k - γ_r enresdowed for any k, where $\gamma_r - 1 + m \le k \le n - 1$.

Proof

Let $G = K_n \odot K_{1,m}$, $n \ge 3$, $m > 2$ be a connected graph on p = n (m +2) vertices . Let D be the γ_r set of G. Let $C_1 = \{u_1, u_2, \ldots, u_l, \ldots, u_r\}$, $1 \le l \le r$ be the vertex set of K_n, $C_2 = \{v_1, v_2, \ldots, v_l, \ldots, v_r\}$, $1 \le l \le r$ be the set of first partition vertices of G and $C_3 = \{a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$, $1 \le l \le mr$ be the set of second partition vertices of G.

Choose the vertex v_1 from the first partition vertices for the γ_r set D of G, then the vertices $a_{11}, a_{12}, \ldots, a_{1m}, u_1$ is dominated, then choose the vertex v_2, for the γ_r set D of G, then the vertices $a_{21}, a_{22}, \ldots, a_{2m}$, u_2 is dominated. Similarly choose the vertex v_3, for the γ_r set D of G, then the vertices $a_{31}, a_{32}, \ldots, a_{3m}$, u_3 is dominated .

Without loss of generality, choose the vertex v_l, $1 \le l \le r$ for the γ_r set D of G , then the vertices $a_{l1}, a_{l2}, \ldots, a_{lm}$, u_l is dominated . Proceeding similarly, choose the vertex v_{r-1} for the γ_r set D of G, then the vertices, $a_{(r-1)1}, a_{(r-1)2}, \ldots, a_{(r-1)m}$, $u_{(r-1)}$ are dominated . Finally choose the vertex v_r, for the γ_r set D of G then the vertices , $a_{r1}, a_{r2}, \ldots, a_{rm}$, u_r is dominated . Thus the γ_r set D is given by $D = \{v_1, v_2, \ldots, v_l, \ldots, v_r\}$, $1 \le l \le r$ and the set $V - D = \{a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}, u_1, u_2, \ldots, u_l, \ldots, u_r\}$. Hence G is k - γ_r enresdowed for any k, where $k = \gamma_r$.

Consider the set D_1 of cardinality k_1, where $k_1 = \gamma_r + 1$. Let $D_1 = \{v_1, v_2, \ldots, v_{l-1}, v_{l+1} \ldots, v_r\} \cup \{a_{li}, a_{lj}\}$, $1 \le i, j \le m$, such that the cardinality of the set of vertices $\{a_{li}, a_{lj}\}$, $1 \le i, j \le m$ is 2, and the set $V - D_1 = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_l, a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{l(i-1)}, a_{l(i+1)} \ldots, a_{l(j-1)}, a_{l(j+1)} \ldots, a_{lm}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$ since $m > 2$, there exist an set of vertices $\{a_{ls}\}$, $1 \le s \le m$, of cardinality $m - 2$ in $V - D_1$, which is not dominated by any vertices { $v_1, v_2, \ldots, v_{l-1}, v_{l+1}, \ldots, v_r$ }. Hence D_1 is not a restrained dominating set of G.

Consider the set D_2 of cardinality k_2, where $k_2 = \gamma_r + 2$. Let $D_2 = \{v_1, v_2, \ldots, v_{l-1}, v_{l+1}, \ldots, v_r\} \cup \{a_{li}, a_{lj}, a_{lp}\}$, $1 \leq i,j,p \leq m$, such that the cardinality of the set $\{a_{li}, a_{lj}, a_{lp}\}$ is 3, and the set $V - D_2 = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_l, a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{l(i-1)}, a_{l(i+1)}, \ldots, a_{l(j-1)}, a_{l(j+1)}, \ldots, a_{l(p-1)}, a_{l(p+1)}, \ldots, a_{lm}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$, then there exist a set of vertices$\{a_{ls}\}$, $1 \leq s \leq m$, of cardinality $m - 3$ in $V - D_2$, which is not dominated by any vertices of the set D_2. Hence D_2 is not a restrained dominating set.

Proceeding similarly, consider a set D_{t_1} of cardinality $k_{t_1} = \gamma_r - 1 + m$, where $D_{t_1} = \{v_1, v_2, \ldots, v_{l-1}, v_{l+1}, \ldots, v_r\} \cup \{a_{l1}, a_{l2}, \ldots, a_{lm}\}$, then the set $V - D_{t_1} = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_l, \ldots, a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{(l1-1)}, a_{(l2-1)}, \ldots, a_{(lm-1)}, a_{(l1+1)}, a_{(l2+1)}, \ldots, a_{(lm+1)}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$. The set D_{t_1} forms the restrained dominating set of cardinality $\gamma_r - 1 + m$, which does not contain the γ_r set D. Therefore G is not k - γ_r enresdowed for any $k_{t_1} = \gamma_r - 1 + m$. Similarly consider a set D_{t_2} of cardinality $k_{t_2} = \gamma_r + m$, $k_{t_2} > k_{t_1}$, where $D_{t_2} = \{v_1, v_2, \ldots, v_{l-1}, v_{l+1}, \ldots, v_r\} \cup \{a_{l1}, a_{l2}, \ldots, a_{lm}, a_{(l1-1)}\}$, then the set, $V - D_{t_2} = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_l, a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{(l2-1)}, \ldots, a_{(lm-1)}, a_{(l1+1)}, a_{(l2+1)}, \ldots, a_{(lm+1)}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$. Thus the set D_{t_2} forms the restrained dominating set of cardinality $\gamma_r + m$, where $k_{t_2} > k_{t_1}$, which does not contain the γ_r set D. Hence G is not k - γ_r enresdowed for any $k_{t_2} = \gamma_r + m$.

Without loss of generality, consider a set $D_{t_3}$of cardinality $k_{t_3} = \gamma_r + m + 1$, where , $k_{t_3} > k_{t_2}$, where $D_{t_3} = \{v_1, v_2, \ldots, v_{l-1}, v_{l+1}, \ldots, v_r\} \cup \{a_{l1}, a_{l2}, \ldots, a_{lm}, a_{(l1-1)}, a_{(l2-1)}\}$, then the set $V - D_{t_3} = \{u_1, u_2, \ldots, u_l, \ldots, u_r, v_l, a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{(l3-1)}, \ldots, a_{(lm-1)}, a_{(l1+1)}, a_{(l2+1)}, \ldots, a_{(lm+1)}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$. Thus the set D_{t_3} forms the restrained dominating set of cardinality $k_{t_3} = \gamma_r + m + 1$, where $k_{t_3} > k_{t_2}$, which does not contain the γ_r set D. Hence G is not k - γ_r enresdowed for any, $k_{t_3} = \gamma_r + m + 1$.

Proceeding similarly, consider a set $D_{t_{q-1}}$ of cardinality $k_{t_{q-1}} = n - 1$, where $k_{t_{q-1}} > k_{t_{q-2}}$, where $D_{t_{q-1}} = \{v_1, v_2, \ldots, v_{l-1}, v_{l+1}, \ldots, v_r, u_1, u_2, \ldots, u_l, \ldots, u_r, \ldots, a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$, then the set $V - D_{t_{q-1}} = \{v_l\}$ which is of cardinality n – 1 and it is not a restrained dominating set of G. Consider a restrained dominating set D_{t_q} of cardinality $k_{t_q} = n\,(m + 2)$, which contains the minimum restrained dominating set D. Thus G is k_{t_q} - γ_r enresdowed for any $k_{t_q} = n\,(m + 2)$.

Definition 2.12

A graph G is said to be Hamiltonian if it contains a spanning cycle. The spanning cycle is called a Hamiltonian cycle of G.[1]

Theorem 2.13

The connected graph $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ is non – Hamiltonian .

Proof

Let $G = K_n \odot K_{1,m}$, for $n \geq 3$ and $m \geq 2$ be a connected graph. Let $C_1 = \{u_1, u_2, \ldots, u_l, \ldots, u_r\}$, $1 \leq l \leq r$ be the vertex set of K_n, $C_2 = \{v_1, v_2, \ldots, v_l, \ldots, v_r\}$, $1 \leq l \leq r$ be the set of first partition vertices of G and $C_3 = \{a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, a_{22}, \ldots, a_{2m}, \ldots, a_{l1}, a_{l2}, \ldots, a_{lm}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rm}\}$, $1 \leq l \leq mr$ be the set of second partition vertices of G.

Consider any vertex $u_l \in V(K_n)$, then $\{u_l, a_{l1}, v_l, a_{lm}, u_l, v_l, a_{l(m-1)}, u_l, a_{l(m-2)}, v_l, a_{l(m-3)}, \ldots, a_{l2}, u_l\}$, $1 \leq l \leq r$ forms a cycle in G . Thus G has n disjoint cycles in G and these cycles are formed at each vertex of K_n . Thus the set, $\{u_1, a_{11}, v_1, a_{1m}, u_1, v_1, a_{1(m-1)}, u_1, a_{1(m-2)}, v_1, a_{1(m-3)}, \ldots, a_{12}, u_1, u_2, a_{21}, v_2, a_{2m}, u_2, v_2, a_{2(m-1)}, u_2, a_{2(m-2)}, v_2, a_{2(m-3)}, \ldots, a_{22}, u_2, u_3, a_{31}, v_3, a_{3m}, u_3, v_3, a_{3(m-1)}, u_3, a_{3(m-2)}, v_3, a_{3(m-3)}, \ldots, a_{32}, u_3, \ldots, u_l, a_{l1}, v_l, a_{lm}, u_l, v_l, a_{l(m-1)}, u_l, a_{l(m-2)}, v_l, a_{l(m-3)}, \ldots, a_{l2}, u_l, \ldots, u_r, a_{r1}, v_r, a_{rm}, u_r, v_r, a_{r(m-1)}, u_r, a_{r(m-2)}, v_r, a_{r(m-3)}, \ldots, a_{r2}, u_r, u_1\}$ forms a cycle in G, in which the vertices are repeated in the cycle. Hence G has no Hamiltonian cycle. Thus G is non – Hamiltonian graph.

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