On α -Kenmotsu Manifolds With Conformal Ricci Soliton

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Abstract: The object of the present paper is to study manifolds admitting the conformal Ricci Solitons and conformal curvature tensor over α -Kenmotsu manifolds. we also study conharmonically Ricci symmetric α -Kenmotsu manifold admitting conformal Ricci soliton.we also proved that a α -Kenmotsu manifold M with projective curvature tensor admitting conformal Ricci soliton in η -Einstein manifold.

Key words: α -Kenmotsu manifold, Conformal Ricci Soliton, Conharmonic Curvature Tensor, Conformal Curvature Tensor, η -Einstein manifold.

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1.Introduction

A $(2n+1)$ -dimensional differentiable manifold M of class C^{∞} is said to have an almost contact structure, if the structural group of its tangent bundle reduces to $U(n)\times 1$, ([7], [8]) equivalently an almost structure is given by a triple (ϕ,ξ,η) satisfying certain conditions (see section 2). Many different types of almost contact structures are defined in the literature (cosymplectic, almost cosymplectic, Sasakian, quasi Sasakian, α -Kenmotsu, [9], [10]).
Recently, the pioneering works of R. Hamilton [1] and G. Perelman [2] towards the solution of the Poincaré conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture.
On a compact Riemannian manifold M, Ricci flow equation is given by

$$\frac{\partial g_{ij}}{\partial t} = -2S \tag{1.1}$$

where g_{ij} is Riemannian metric, and S is Ricci curvature tensor. Ricci soliton emerges as limit of the solutions of Ricci flow. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling.
A Ricci soliton (g,X,λ) on Riemannian manifold M ([3], [4]) is given by

$$£_X g + 2S + 2\lambda g = 0 \tag{1.2}$$

where $£_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is vector field and λ is real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0,$ and $\lambda > 0$ respectively.

In 2005, A.E. Fischer [5] introduced a new concept called conformal Ricci flow is a variation of Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equation are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equation are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pq, \text{and } r(g) = -1, \tag{1.3}$$

where p is a non-dynamical scalar field (time dependent scalar field), $r(g)$ is the scaler curvature of the manifold and n is the dimension of manifold M.

In 2015, N. Basu and A. Bhattacharya [6] introduced the notion of conformal Ricci soliton and the equation as follows

$$£_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g. \tag{1.4}$$

Where λ is constant and $£_X$ is the Lie derivative.

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold with Ricci tensor is proportional to the metric. If M is the underlying n-dimensional manifold and g is its metric tensor then the Einstein condition means that

$$S(X,Y) = \lambda g(X,Y),$$

for some constant λ, where S denotes the Ricci tensor of g. Einstein manifolds with $\lambda = 0$ are called Ricci-flat manifolds.

A trans-Sasakian manifold M^n is said to be η Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

The main purpose of this paper is to investigate the class of almost contact metric manifolds which are called α-Kenmotsu manifolds. These manifolds appear for the first time in (see [11]), where they have been locally classified.

The present paper is organized as: Section 2 is Introductory. In section 3, we obtain some important condition of Lorentzian α-Kenmotsu manifolds admitting conformal Ricci soliton and $R(\xi, X)\widetilde{C} = 0$. In section 4, we have Lorentzian α-Kenmotsu manifolds admitting conformal Ricci soliton and $K.(\xi, X)S = 0$. In section 5, we introduced Lorentzian α-Kenmotsu manifoldsadmitting conformal Ricci soliton and $P(\xi, X)\widetilde{C} = 0$.

2. Preliminary

An almost contact manifolds is an odd-dimensional manifold M^{2n+1} with (1,1) tensor field ϕ has contant rank 2n, a vector field ξ, is called characteristic vector field, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \eta \circ \varphi = 0, \ \ \varphi\xi = 0. \tag{2.2}$$

An almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is said to be normal when tensor field $N = [\phi, \phi] + 2dn \otimes \xi$ vanishes identicaly, $[\phi, \phi]$ denoting the Nijenhuis tensor ϕ.
Almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

for any vector fields X, Y on M^{2n+1}. This metric g is called compatible metric and the manifold M^{2n+1} together with the structure $(M^{2n+1}, \phi, \xi, \eta, g)$ is called an almost contact metric manifold.
An almost contact metric manifold M^{2n+1} is said to be almost α-Kenmotsu manifold if

$$d\eta = 0 \ \text{ and } \ d\phi = 2\alpha\eta\Lambda\phi$$

α being a non-zero real constant.
Now we have

$$\nabla_X \xi = -\alpha^2 \phi X, \tag{2.5}$$

$$(\nabla_X \eta)Y = \alpha g(X, Y) - \eta(X)\eta(Y)], \tag{2.6}$$

where ∇ denotes the operator of covariant differentiation with respect to the Kenmotsu metric g on M.
On an $\alpha-$Kenmotsu manifold M the following relations hold:

$$R(X, Y)\xi = \alpha^2[-\eta(Y)X + \eta(X)Y], \tag{2.7}$$

$$R(\xi, X)Y = \alpha^2[-g(X, Y)\xi + \eta(Y)X], \tag{2.8}$$

$$S(X, \xi) = -\alpha^2(n-1)\eta(X), \tag{2.9}$$

$$Q\xi = -\alpha^2(n-1)\xi,$$

$$S(\xi, \xi) = -\alpha^2(n-1), \tag{2.10}$$

where α is some constant, R is the Riemannian curvature, S is Ricci tensor and Q is the Ricci operator given by $S(X,Y)=g(QX,Y)$ for all $X,Y\in\chi(M)$.
Now from definition of Lie derivative we have

$$(\pounds_{\xi}g)(X,Y)=(\nabla_{\xi}g)(X,Y)+g(\alpha\varphi X,Y)+g(X,\alpha\varphi Y)=0, \tag{2.11}$$

Applying (2.11) in (1.4) we get

$$S(X,Y)=\frac{1}{2}[2\lambda-(p+\frac{2}{n})]g(X,Y)=Ag(X,Y), \tag{2.12}$$

where

$$A=\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right].$$

Since $S(X,Y)=g(QX,Y)$ for the Ricci operator Q, we have

$$g(QX,Y)=Ag(X,Y) \tag{2.13}$$

i.e.,

$$QX=AX,.\ \forall X \tag{2.14}$$

Also

$$S(Y,\xi)=A\eta(Y), S(\xi,\xi)=A \quad Q\xi=A\xi. \tag{2.15}$$

If we put $X=Y=e_i$ in (2.12), where $\{e_i\}$ is orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i, we get

$$r(g)=An$$

As $r=-1$, we have

$$A=-\frac{1}{n} \tag{2.16}$$

3. $\alpha-$Kenmotsu manifold admittingconformal Ricci soliton and $R(\xi,X).\widetilde{C}=0$

Let M be an $(2n+1)$-dimensional $\alpha-$Kenmotsu manifold admitting a conformal Ricci Soliton (g,V,λ). The conformal curvature tensor $\widetilde{C}$ on M is defined by:

$$\widetilde{C}(X,Y)Z=R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX \tag{3.1}$$

$$-g(X,Z)QY]+\frac{r}{2n(n-1)}[g(Y,Z)X-g(X,Z)Y],$$

where r is scalar curvature.
Now we prove the following theorem:

Theorem 3.1: *If an $\alpha-$ Kenmotsu manifold admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. $R(\xi,X).\widetilde{C}=0$, then the manifold is η-Einstein manifold where $\widetilde{C}$ is conformal curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

Proof: Let M be α-Kenmotsu manifold admitting a conformal Ricci soliton (g,V,λ). So we have $r=-1$.
After putting $r=-1$ and $Z=\xi$ in (3.1) we have

$$\widetilde{C}(X,Y)\xi = R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] - \frac{1}{2n(n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \tag{3.2}$$

Using (2.2), (2.4), (2.11) and (2.12) in (3.2) we get

$$\widetilde{C}(X,Y)\xi = \alpha^2[-\eta(Y)X+\eta(X)Y] - \frac{1}{2n-1}[An(Y)X - An(X)Y + \eta(Y)(AX) - \eta(X)(AY)] - \frac{1}{2n(n-1)}[\eta(Y)X - \eta(X)Y]. \tag{3.3}$$

Using (3.1) and after simplification we obtain

$$\widetilde{C}(X,Y)\xi = \left[\alpha^2 + \frac{2A}{2n-1} + \frac{1}{2n(n-1)}\right][\eta(X)Y - \eta(Y)X]. \tag{3.4}$$

Considering

$$B = \alpha^2 + \frac{2A}{2n-1} + \frac{1}{2n(n-1)},$$

Now (3.4) becomes

$$\widetilde{C}(X,Y)\xi = B[-\eta(Y)X + \eta(X)Y], \tag{3.5}$$

and

$$g(\widetilde{C}(X,Y)\xi, Z) = B[-\eta(Y)g(X,Z) + \eta(X)g(Y,Z)], \tag{3.6}$$

which implies

$$\eta(\widetilde{C}(X,Y)Z) = B[-\eta(Y)g(X,Z) + \eta(X)g(Y,Z)]. \tag{3.7}$$

Now we consider that the $\alpha-$Kenmotsu manifold admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. $R(\xi,X).\widetilde{C}=0$ holds in M, which implies

$$R(\xi,X)(\widetilde{C}(Y,Z)W)-\widetilde{C}(R(\xi,X)Y,Z)W)-\widetilde{C}(Y,R(\xi,X)Z)W)$$
$$-\widetilde{C}(Y,Z)R(\xi,X)W=0, \tag{3.8}$$

for all vector fields X,Y,Z,W on M.
Using (2.7) in (3.8) and putting $W=\xi$ we get

$$-g(X,\widetilde{C}(Y,Z)\xi)\xi+\eta(\widetilde{C}(Y,Z)\xi)X+g(X,Y)\widetilde{C}(\xi,Z)\xi$$
$$-\eta(Y)\widetilde{C}(X,Z)\xi)+g(X,Z)\widetilde{C}(Y,\xi)\xi-\eta(Z)\widetilde{C}(Y,X)\xi$$
$$+g(X,\xi)\widetilde{C}(Y,Z)\xi-\eta(\xi)\widetilde{C}(Y,Z)X=0. \tag{3.9}$$

Taking inner product with ξ in (3.9) and using (2.1) we obtain

$$-g(X,\widetilde{C}(Y,Z)\xi)+g(X,Y)\eta(\widetilde{C}(\xi,Z)\xi)-\eta(Y)\eta(\widetilde{C}(X,Z)\xi)$$
$$+g(X,Z)\eta(\widetilde{C}(Y,\xi)\xi)-\eta(Z)\eta(\widetilde{C}(Y,X)\xi)+\eta(X)\eta(\widetilde{C}(Y,Z)\xi)`$$
$$-\eta(\widetilde{C}(Y,Z)X)=0. \tag{3.10}$$

Using (3.5) in (3.10) we have

$$B\eta(Z)g(X,Y)-B\eta(Y)g(X,Z)-\eta(\widetilde{C}(Y,Z)X)=0. \tag{3.11}$$

Putting $Z=\xi$ in (3.11) and using (2.1) we get

$$Bg(X,Y)-B\eta(Y)\eta(X)-\eta(\widetilde{C}(Y,\xi)X)=0. \tag{3.12}$$

Now from (3.1)

$$\widetilde{C}(Y,\xi)X=R(Y,\xi)X-\frac{1}{2n-1}[S(\xi,X)Y-S(Y,X)\xi+g(\xi,X)QY$$
$$-g(Y,X)Q\xi]-\frac{1}{2n(n-1)}[g(\xi,X)Y-g(Y,X)\xi]. \tag{3.13}$$

Taking inner product with ξ and using (2.1), (2.8), in (3.13), we get

$$\eta(\widetilde{C}(Y,\xi)X)=-\alpha^2\eta(X)\eta(Y)+\alpha^2 g(X,Y)$$
$$-\frac{A}{2n-1}\eta(X)\eta(Y)+\frac{1}{2n-1}S(X,Y)$$
$$-\frac{A}{2n-1}\eta(X)\eta(Y)+\frac{A}{2n-1}g(X,Y)$$
$$-\frac{1}{2n(n-1)}\eta(X)\eta(Y)+\frac{1}{2n(n-1)}g(X,Y). \tag{3.14}$$

After putting (3.14) in (3.12) the equation reduces to

$$Bg(X,Y)-B\eta(Y)\eta(X)+\alpha^2\eta(X)\eta(Y)-\alpha^2 g(X,Y)+\frac{A}{2n-1}\eta(X)\eta(Y) \tag{3.15}$$
$$-\frac{1}{2n-1}S(X,Y)+\frac{A}{2n-1}\eta(X)\eta(Y)-\frac{A}{2n-1}g(X,Y)+\frac{1}{2n(n-1)}\eta(X)\eta(Y)$$
$$-\frac{1}{2n(n-1)}g(X,Y)=0.$$

Simplifying (3.15) we have

$$g(X,Y)\left[B-\alpha^2-\frac{A}{2n-1}-\frac{1}{2n(n-1)}\right]+\eta(X)\eta(Y)\left[-B+\alpha^2+\frac{2A}{2n-1}+\frac{1}{2n(n-1)}\right]$$
$$-\frac{1}{2n-1}S(X,Y)=0. \tag{3.16}$$

which can be written in the form

$$S(X,Y)=\rho g(X,Y)+\sigma\eta(X)\eta(Y), \tag{3.17}$$

where

$$\rho=(2n-1)\left[B-\alpha^2-\frac{A}{2n-1}-\frac{1}{2n(n-1)}\right]$$

and

$$\sigma=(2n-1)\left[-B+\alpha^2+\frac{2A}{2n-1}+\frac{1}{2n(n-1)}\right].$$

So from (3.17) we conclude that the manifold becomes η-Einstein manifold.

4. $\alpha-$Kenmotsu manifold admitting conformal Ricci soliton and $K(\xi,X).S=0$

Let M be an (2n+1)-dimensional $\alpha-$Kenmotsu manifolds admitting a conformal Ricci soliton (g,V,λ). The conharmonic curvature tensor K on M is defined by

$$K(X,Y)Z=R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY], \tag{4.1}$$

for all $X,Y,Z\in\chi(M)$, R is the curvature tensor and Q is the Ricci operator.
Now we prove the following theorem:

Theorem 4.1: *If a $\alpha-$Kenmotsu manifolds admits a conformal Ricci soliton and the manifold is conharmonically Ricci symmetric i.e. $K(\xi,X).S=0$ then the Ricci operator Q satisfies the* *quadratic equation* $FQ^2 - Q + D = 0$ *for all* $X \in \chi(M)$ *where* F *,* D *are constants,* K *is conharmonic curvature tensor and* S *is a Ricci tensor.*

Proof: Let M be an $\alpha-$Kenmotsu manifold admitting a conformal Ricci soliton (g, V, λ). From (4.1) we can write

$$K(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n-1}[S(X,Y)\xi - S(\xi,Y)X + g(X,Y)Q\xi - g(\xi,Y)QX]. \tag{4.2}$$

Using (2.7), (2.14) in (4.2) we have

$$K(\xi, X)Y = \alpha^2[-g(X,Y)\xi + \eta(Y)X] - \frac{1}{2n-1}[S(X,Y)\xi - A\eta(Y)X - Ag(X,Y)\xi + \eta(Y)QX]. \tag{4.3}$$

Similarly from (4.2) we get

$$K(\xi, X)Z = R(\xi, X)Z - \frac{1}{2n-1}[S(X,Z)\xi - S(\xi,Z)X + g(X,Z)Q\xi - g(\xi,Z)QX] \tag{4.4}$$

,

which gives

$$K(\xi, X)Z = \alpha^2[-g(X,Z)\xi + \eta(Z)X] - \frac{1}{2n-1}[S(X,Z)\xi - A\eta(Z)X - Ag(X,Z)\xi + \eta(Z)QX].$$

Now we consider that the tensor derivative of S by $K(\xi, X)$ is zero, i.e, $K(\xi, X).S = 0$, then the $\alpha-$Kenmotsu manifoldadmitting conformal Ricci soliton is conharmonically Ricci symmetric, It gives

$$S(K(\xi, X)Y, Z) + S(Y, K(\xi, X)Z) = 0. \tag{4.5}$$

Using (4.3) and (4.4) in (4.5) we get

$$S(-\alpha^2 g(X,Y)\xi + \alpha^2\eta(Y)X - \frac{1}{2n-1}S(X,Y)\xi + \frac{A}{2n-1}\eta(Y)X - \frac{A}{2n-1}g(X,Y)\xi + \frac{\eta(Y)}{2n-1}QX, Z) + S(-\alpha^2 g(X,Z)\xi + \alpha^2\eta(Z)X - \frac{1}{2n-1}S(X,Z)\xi + \frac{A}{2n-1}\eta(Z)X - \frac{A}{2n-1}g(X,Z)\xi + \frac{\eta(Z)}{2n-1}QX, Y) = 0. \tag{4.6}$$

Putting $Z = \xi$ and using (2.1), (2.14) in (4.6) we get

$$\left(-\frac{A^2}{2n-1} - A\alpha^2\right)g(X,Y) + \alpha^2 S(X,Y) + \frac{1}{2n-1}S(QX,Y) = 0 \tag{4.7}$$

which implies

$$Eg(X,Y)=\frac{1}{2n-1}S(QX,Y)=-\alpha^2 S(X,Y), \tag{4.8}$$

where $E=-\frac{A^2}{2n-1}-A\alpha^2$.
From (4.8) we can write

$$S(X,Y)=-Dg(X,Y)-\frac{1}{\alpha^2(2n-1)}S(QX,Y), \tag{4.9}$$

where $D=-\frac{1}{\alpha^2}E$, which implies

$$QX=-DX+FQ^2 \qquad \forall X\in\chi(M), \tag{4.10}$$

where $F=-\frac{1}{\alpha^2(2n-1)}$, i.e.

$$FQ^2-Q+D=0 \tag{4.11}$$

Hence the theorem follows:

5. $\alpha-$Kenmotsu manifold admitting conformal Ricci soliton and $P(\xi,X).\widetilde{C}=0$

The Weyl projective curvature tensor P on M is given as

$$P(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-S(X,Z)Y]. \tag{5.1}$$

Now we have the following theorem:

Theorem 5.1 *If a $\alpha-$Kenmotsu manifold M admits conformal Ricci soliton and $P(\xi,X).\widetilde{C}=0$ holds, then the manifold becomes $\eta-$Einstein manifold, where P is projective curvature tensor and $\widetilde{C}$ is conformal curvature tensor.*

Proof: We know from (3.1) that

$$P(\xi,X)Y=R(\xi,X)Y-\frac{1}{2n-1}S(X,Y)\xi-S(\xi,Y)X+g(X,Y)Q\xi \tag{5.2}$$

$$-g(\xi,Y)QX]-\frac{1}{2n(n-1)}[g(X,Y)\xi-g(\xi,Y)X],$$

since for conformal Ricci soliton the scalar curvature $r=-1$.
From (2.7), (2.14) and taking inner product with product with ξ on (5.2) we have

$$\eta(\widetilde{C}(\xi,X)Y)=-\alpha^2 g(X,Y)\eta(\xi)+\alpha^2\eta(Y)\eta(X)-\frac{1}{2n-1}S(X,Y)\eta(\xi) \tag{5.3}$$
$$+\frac{A}{2n-1}\eta(Y)\eta(X)-\frac{A}{2n-1}\eta(\xi)g(X,Y)+\frac{1}{2n-1}\eta(Y)\eta(QX)$$
$$-\frac{1}{2n(n-1)}[g(X,Y)\xi-\eta(Y)\eta(X)],$$

which implies

$$\eta(\widetilde{C}(\xi,X)Y)=g(X,Y)\left[-\frac{A}{2n-1}-\alpha^2-\frac{1}{2n(n-1)}\right]+\eta(Y)\eta(X)\left[\frac{2A}{2n-1}+\alpha^2+\frac{1}{2n(n-1)}\right]$$
$$-\frac{1}{2n-1}S(X,Y)=Fg(X,Y)+G\eta(Y)\eta(X)+TS(X,Y),$$

where

$$F=-\frac{A}{2n-1}-\alpha^2+\frac{1}{2n(n-1)},$$

$$G=-\frac{2A}{2n-1}+\alpha^2+\frac{1}{2n(n-1)},$$

and

$$T=-\frac{1}{2n-1},$$

also

$$\eta(\widetilde{C}(X,Y)\xi)=B[\eta(Y)\eta(X)+\eta(X)\eta(Y)]=0 \tag{5.4}$$

and

$$\eta(\widetilde{C}(Y,\xi)\xi)=B[\eta(Y)\eta(\xi)-\eta(\xi)\eta(Y)]=0. \tag{5.5}$$

Now

$$P(\xi,X)Y=R(\xi,X)Y-\frac{1}{2n}[S(X,Y)\xi-S(\xi,Y)X]. \tag{5.6}$$

Using (2.7), (2.14) in (5.6) we get

$$P(\xi,X)Y=\alpha^2[-g(X,Y)\xi+\eta(Y)X]-\frac{1}{2n}[S(X,Y)\xi-A\eta(Y)X]. \tag{5.7}$$

Here we consider that the tensor derivative of $\widetilde{C}$ by $P(\xi,X)$ is zero i.e. conformally symmetric with respect to projective curvature tensor i.e. $P(\xi,X).\widetilde{C}=0$ holds.So

$$P(\xi,X)\widetilde{C}(Y,Z)W-\widetilde{C}(P(\xi,X)Y,Z)W-\widetilde{C}(Y,P(\xi,X)Z)W-\widetilde{C}(Y,Z)P(\xi,X)W=0, \tag{5.8}$$

for all vector fields X,Y,Z,W on M.

Using (5.7) in (5.8) and putting $W=\xi$ we have

$$\begin{aligned}
&-\alpha^2 g(X,\widetilde{C}(Y,Z)\xi)\xi+\alpha^2\eta(\widetilde{C}(Y,Z)\xi)X \\
&-\frac{1}{2n}S(X,\widetilde{C}(Y,Z)\xi)\xi+\frac{A}{2n}\eta(\widetilde{C}(Y,Z)\xi)X+\alpha^2 g(X,Y)\widetilde{C}(\xi,Z)\xi \\
&-\alpha^2\eta(Y)\widetilde{C}(X,Z)\xi+\frac{1}{2n}S(X,Y)\widetilde{C}(\xi,Z)\xi-\frac{A}{2n}\eta(Y)\widetilde{C}(X,Z)\xi \\
&+\alpha^2 g(X,Z)\widetilde{C}(Y,\xi)\xi+\alpha^2\eta(Z)\widetilde{C}(Y,X)\xi+\frac{1}{2n}S(X,Z)\widetilde{C}(Y,\xi)\xi \\
&-\frac{A}{2n}\eta(Z)\widetilde{C}(Y,X)\xi+\alpha^2 g(X,\xi)\widetilde{C}(Y,Z)\xi-\alpha^2\eta(\xi)\widetilde{C}(Y,Z)X \\
&+\frac{1}{2n}S(X,\xi)\widetilde{C}(Y,Z)\xi-\frac{A}{2n}\eta(\xi)\widetilde{C}(Y,Z)X=0.
\end{aligned} \tag{5.9}$$

Taking inner product with ξ on (5.9) we get

$$-\alpha^2 g(X,\widetilde{C}(Y,Z)\xi)-\frac{1}{2n}S(X,\widetilde{C}(Y,Z)\xi)=0. \tag{5.10}$$

From (3.2) and (5.10) we have

$$\alpha^2 B\eta(Z)g(X,Y)-\alpha^2\eta(Y)Bg(X,Z)-\frac{B}{2n}\eta(Z)S(X,Y)+\frac{B}{2n}\eta(Y)S(X,Z)=0. \tag{5.11}$$

Putting $Z=\xi$ in (5.11) and using (2.1), (2.14) we obtain

$$-\alpha^2 Bg(X,Y)-B\alpha^2\eta(Y)\eta(X)-\frac{B}{2n}S(X,Y)+\frac{AB}{2n}\eta(Y)\eta(X)=0, \tag{5.12}$$

which implies

$$S(X,Y)=(-2n\alpha^2)g(X,Y)+2n\left(\frac{A}{2n}-\alpha^2\right)\eta(X)\eta(Y) \tag{5.13}$$

So the manifold becomes η-Einstein manifold.

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